Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: ______________________________

13.5 Curl and Divergence

Definition. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, what is the curl of $\mathbf{F}$?

Example. If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find $\operatorname{curl}\mathbf{F}$.

Theorem.

(a) Show that the curl of a gradient vector field is $\mathbf{0}$.

(b) What can we conclude about conservative vector fields?

Example. Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

Theorem. If $\operatorname{curl} \mathbf{F} = 0$, is F a conservative vector field?

Example.

(a) Show that $\mathbf{F}(x, y, z) = y^2 z^3\, \mathbf{i} + 2xyz^3\, \mathbf{j} + 3xy^2 z^2\, \mathbf{k}$ is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Definition. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$, and $\partial P/\partial x, \partial Q/\partial y$, and $\partial R/\partial z$ exist, what is the divergence of $\mathbf{F}$?

Example. If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find $\operatorname{div}\mathbf{F}$.

Theorem. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, show that $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.

Example. Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$.

Theorem. Use the curl and divergence operators to give two ways to rewrite Green's Theorem in vector form:

(a) $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$

(b) $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint\limits_D \operatorname{div} \mathbf{F}(x, y) \, dA$