Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

Rotation Flow in/out

13.5 Curl and Divergence

Definition. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, what is the curl of $\mathbf{F}$?

$$\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

- To remember this, use the vector differential operator ∇ ("del")
- $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ ← This is an operator
- Then

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \text{curl } \vec{F}$$

Rmk: For 2D vector fields, think about $F = \langle P, Q \rangle$ as $F = \langle P, Q, 0 \rangle$.

Example. If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find $\operatorname{curl}\mathbf{F}$.

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\vec{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\vec{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\vec{k}$$

$$= (-2y - xy)\,\vec{i} - (0 - x)\,\vec{j} + (yz - 0)\,\vec{k}$$

$$= -y(2+x)\,\vec{i} + x\,\vec{j} + yz\,\vec{k}$$

Theorem.

(a) Show that the curl of a gradient vector field is $\mathbf{0}$.

(b) What can we conclude about conservative vector fields?

(a) $\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$ by Clairaut's Theorem.

(b) This means that any conservative vector field $\vec{F}$ has $\operatorname{curl} \vec{F} = \vec{0}$.

Example. Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

We showed that $\operatorname{curl} \vec{F} = -y(2+x)\vec{i} + x\vec{j} + yz\vec{k}$.

This is nonzero, so $\vec{F}$ is not conservative.

Theorem. If curl $\mathbf{F} = 0$, is F a conservative vector field?

This is true if $\vec{F}$ is a vector field defined on a simply-connected domain.

(In particular, this is true if $\vec{F}$ is defined on all of $\mathbb{R}^3$)

Example.

(a) Show that $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$ is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

(a) $$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\vec{i} - (3y^2z^2 - 3y^2z^2)\,\vec{j} + (2yz^3 - 2yz^3)\,\vec{k}$$

$$= \vec{0}$$

Since curl $\vec{F} = 0$ and the domain of $\vec{F}$ is $\mathbb{R}^3$, $\vec{F}$ is conservative.

(b) EQ1 $f_x(x,y,z) = y^2z^3$

EQ2 $f_y(x,y,z) = 2xyz^3$

EQ3 $f_z(x,y,z) = 3xy^2z^2$

Integrating EQ1 w.r.t. x,

$$\boxed{f(x,y,z) = xy^2z^3 + g(y,z)}$$

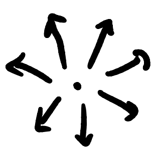
- $f_y(x,y,z) = 2xyz^3 + g_y(y,z)$
- $g_y(y,z) = 0$ by EQ2, which implies $g(y,z) = h(z)$
- $f_z(x,y,z) = 3xy^2z^2 + h'(z)$
- By EQ3, $h'(z) = 0$
- Conclude : $\boxed{f(x,y,z) = xy^2z^3 + K}$

Definition. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$, and $\partial P/\partial x, \partial Q/\partial y$, and $\partial R/\partial z$ exist, what is the divergence of $\mathbf{F}$?

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\ &= \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle \\ &= \nabla \cdot \vec{F} \end{aligned}$$

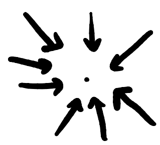
⚠ curl $\vec{F}$ is a vector field, but div $\vec{F}$ is a scalar field.

Think about air hockey.



$\nabla \cdot \vec{F} > 0$

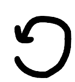
"Source"



$\nabla \cdot \vec{F} < 0$

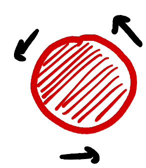
" Sink "

neg. curl



pos. curl

$\nabla \cdot \vec{F}$ measures the sink/source behavior of a point



$\nabla \times \vec{F}$ controls the spinning about the center

Example. If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find $\operatorname{div} \mathbf{F}$.

$$\begin{aligned} \operatorname{div} \vec{F} = \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) \\ &= z + xz \end{aligned}$$

Theorem. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, show that $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.

$$\operatorname{div} \operatorname{curl} \vec{F} = \nabla \cdot (\nabla \times \vec{F})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial z \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y}$$

$$= 0$$

The terms cancel in pairs by Clairaut's theorem

Example. Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$.

If $\vec{F} = \operatorname{curl} \vec{G}$ for some $\vec{G}$, then we would have

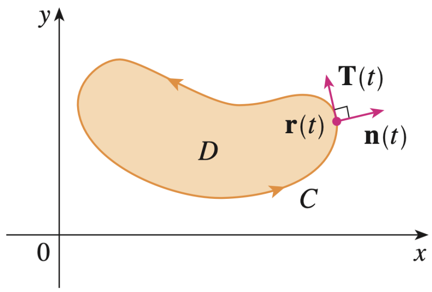
$\operatorname{div} \vec{F} = \operatorname{div} \operatorname{curl} \vec{G} = 0$

But we showed $\operatorname{div} \vec{F} = z + xz \neq 0$

Theorem. Use the curl and divergence operators to give two ways to rewrite Green's Theorem in vector form:

(a) $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA$ ← This expresses $\int_C \vec{F} \cdot d\vec{r}$ as the double integral of the vertical component of curl $\vec{F}$ over the region D enclosed by C.

(b) $\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$ ← The line integral of the normal component of $\vec{F}$ along C is equal to the double integral of the divergence of $\vec{F}$ over D.



(a) Regarding $\vec{F}$ as a vector field on $\mathbb{R}^3$ with 0 $\vec{k}$-component.

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$$\Rightarrow (\operatorname{curl} \vec{F}) \cdot \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Conclude:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy \overset{\text{Green's Theorem}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \overset{\text{By the above}}{=} \iint_D \operatorname{curl} \vec{F} \cdot \vec{k} \, dA$$

(b)

$$\vec{n}(t) = \frac{y'(t)}{|\vec{r}\,'(t)|}\vec{i} - \frac{x'(t)}{|\vec{r}\,'(t)|}\vec{j}$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_a^b (\vec{F}\cdot\vec{n})(t)\,|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \left[\frac{P(x(t),y(t))\,y'(t)}{|\vec{r}\,'(t)|} - \frac{Q(x(t),y(t))\,x'(t)}{|\vec{r}\,'(t)|}\right]|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b P(x(t),y(t))\,y'(t) - Q(x(t),y(t))\,x'(t)\,dt$$

Green's Thm

$$= \int_C P\,dy - Q\,dx = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,dA$$

$$= \iint_D \operatorname{div} F(x,y)\,dA$$