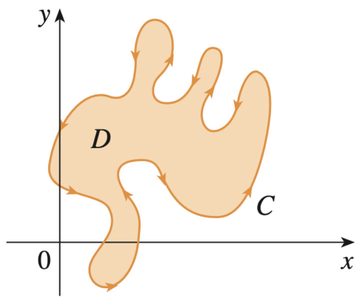
Lecture Notes
Math 2400 - Calculus III
Spring 2024

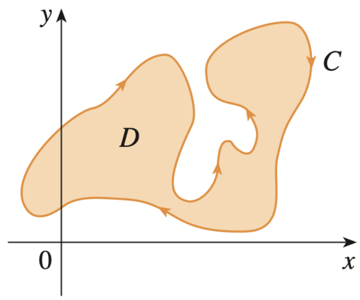
Name: ______________________________

13.4 Green's Theorem

Question. What is the idea of Green's theorem? What is the positive orientation of a simple closed curve C?

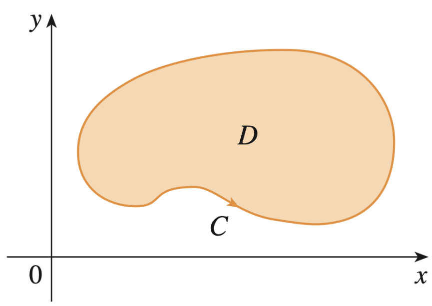


(a) Positive orientation

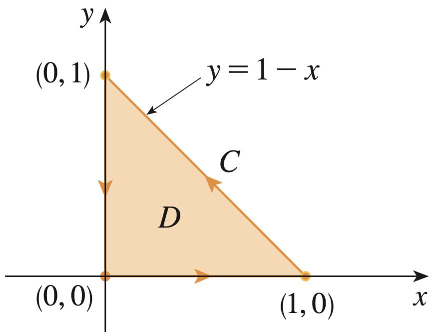


(b) Negative orientation

Theorem (Green's Theorem)**.** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, where P and Q have continuous partial derivatives on an open region that contains D, how can we evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$?



Example. Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

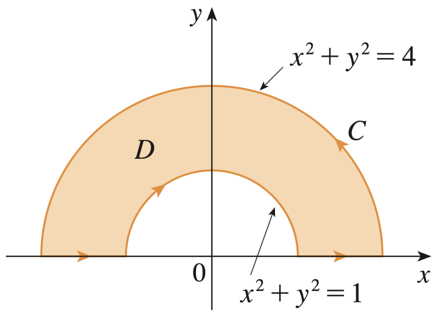


Example. Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

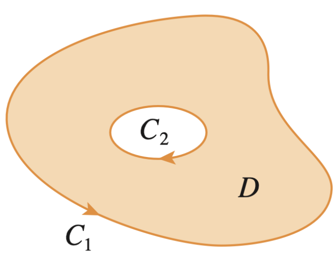
Question. How can we use Green's Theorem to compute the area of a region D?

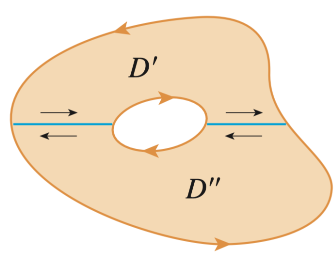
Example. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example. Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



Question. How can we extend Green's Theorem to apply to regions that are not simply-connected?





Example. If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.