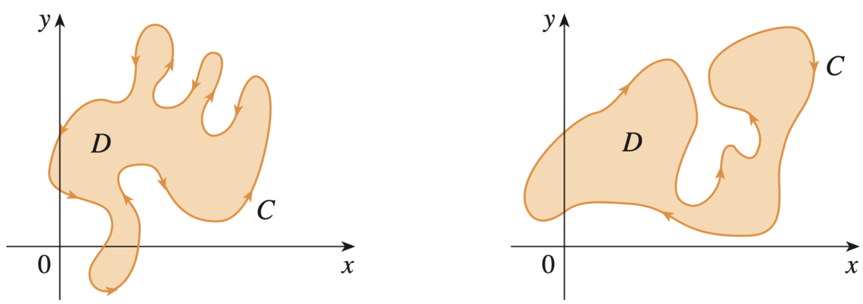
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

13.4 Green's Theorem

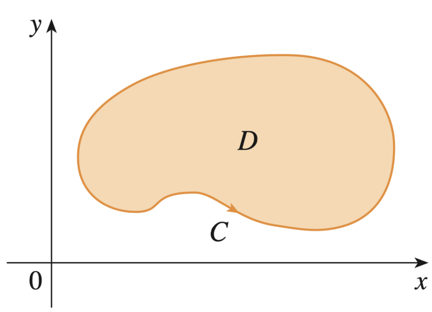
Question. What is the idea of Green's theorem? What is the positive orientation of a simple closed curve C?



(a) Positive orientation (b) Negative orientation

- Green's Theorem relates a line integral around a simple closed C and a double integral over the region D bounded by C.
- Positive orientation is a single counterclockwise traversal (think about the right hand rule)

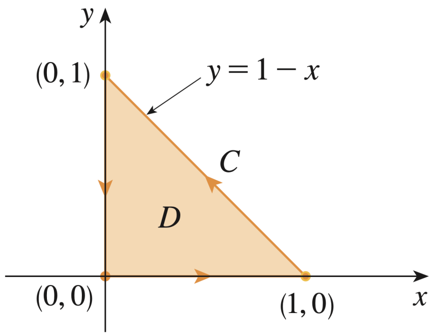
Theorem (Green's Theorem)**.** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, where P and Q have continuous partial derivatives on an open region that contains D, how can we evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$?



$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

Note: sometimes written as $\int_{\partial D} P\,dx + Q\,dy$

Example. Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.



Rmk: We could set this up as a sum of three line integrals, but Green's Thm is easier.

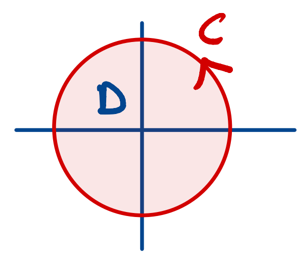
$$\int_C x^4 dx + xy\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \int_0^1 \int_0^{1-x} y - 0\;dy\,dx = \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = \left[-\tfrac{1}{6}(1-x)^3\right]_{x=0}^{x=1} = \frac{1}{6}$$

Example. Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

The region D bounded by C is the disk $x^2+y^2 \leq 9$



$$\oint_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy$$

$$= \iint_D \frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin x}\right)\,dA$$

$$= \iint_D 7 - 3\;dA$$

Convert to Polar

$$= \int_0^{2\pi}\int_0^3 4\,r\,dr\,d\theta$$

$$= 4 \cdot \int_0^{2\pi} d\theta \cdot \int_0^3 r\,dr$$

$$= 36\pi$$

Rmk: Try to compute this without using Green's Thm.

Question. How can we use Green's Theorem to compute the area of a region D?

- The area of D is $\iint_D 1\, dA$
- Idea: find P and Q so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ so that

  $\iint_D 1\, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$ and apply Green's theorem

- Several possibilities:

$P(x,y) = 0$, $Q(x,y) = x$

$P(x,y) = -y$, $Q(x,y) = 0$

$P(x,y) = -\frac{1}{2}y$, $Q(x,y) = \frac{1}{2}x$

- Hence

$$A = \iint_D 1\, dA = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C x\, dy - y\, dx$$

Example. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

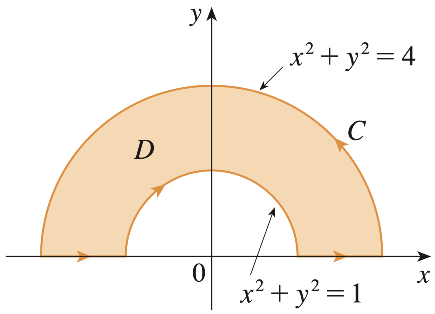
- A parametrization for the ellipse is $\vec{r}(t) = \langle a\cos t, b\sin t\rangle$ for $0 \leq t \leq 2\pi$
- By the above,

$$A = \frac{1}{2}\oint_C x\, dy - y\, dx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\, dt - (b\sin t)(-a\sin t)\, dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \boxed{\pi ab}$$

Example. Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.



D is easily described in polar coordinates by

$D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

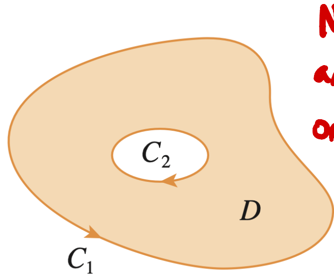
$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\,dA$$

$$= \iint_D 3y - 2y\,dA = \iint_D y\,dA = \int_0^{\pi}\int_1^2 r\sin\theta\cdot r\,dr\,d\theta$$

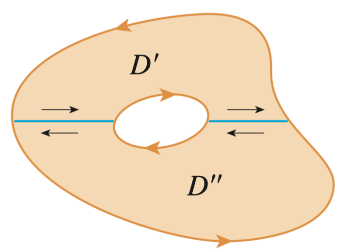
$$= \int_0^{\pi}\sin\theta\,d\theta\cdot\int_1^2 r^2\,dr = \frac{14}{3}$$

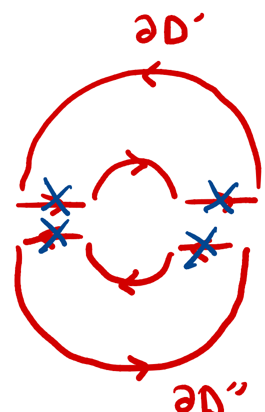
Question. How can we extend Green's Theorem to apply to regions that are not simply-connected?

The boundary C of the region D consists of two curves C_1 and C_2



Note that C_1 and C_2 are oriented so that D is always on the left.





$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

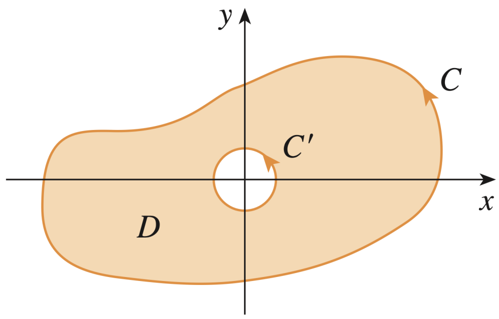
$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

$$= \int_C P\,dx + Q\,dy$$

The line integrals along the common boundary are in opposite directions, so they cancel.

Example. If $\mathbf{F}(x,y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2+y^2)$, show that $\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.



Idea: given an arbitrary curve C, we will use Green's Theorem to show $\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$, where we have a parametrization for C′.

- Consider a positively-oriented circle C′ that lies inside of C. C′ is centered at the origin and has radius a.
- Let D be the region bounded by C and C′. The boundary of D is $C \cup (-C')$ ← D needs to be on the left

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_D \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} \right] dA = 0$$

Hence $\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy \Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$

We can easily compute the last integral, since C′ is parametrized by $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$ for $0 \leq t \leq 2\pi$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$