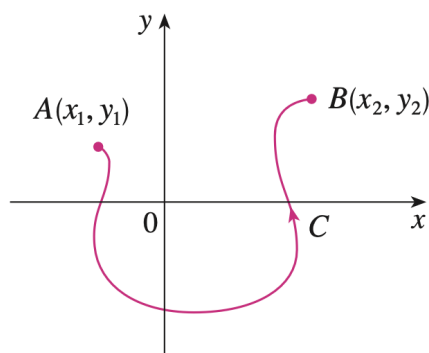
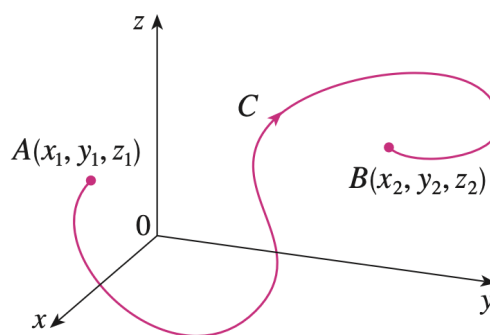


13.3 The Fundamental Theorem for Line Integrals

Theorem (Fundamental Theorem for Line Integrals). Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C . What is $\int_C \nabla f \cdot d\mathbf{r}$?



(a)

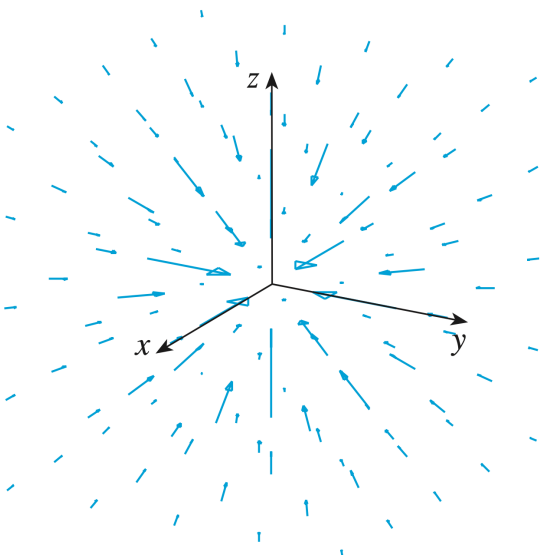


(b)

Proof.

□

Example. Find the work done by gravity in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a smooth curve C .



- The magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

- Place the object with mass M at the origin of $\mathbb{R}^3$ and the object with mass m at $\mathbf{x} = \langle x, y, z \rangle$. The gravitational force acts toward the origin, in the direction of the unit vector $-\frac{\mathbf{x}}{|\mathbf{x}|}$.
- Hence, the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z \rangle$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

- Since $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$, we can write $\mathbf{F}$ in terms of its component functions as

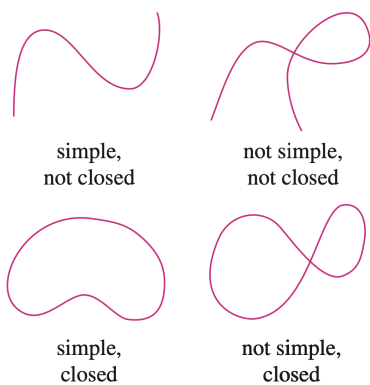
$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

Definition. If $\mathbf{F}$ is a continuous vector field with domain D , what does it mean for $\int_C \mathbf{F} \cdot d\mathbf{r}$ to be independent of path?

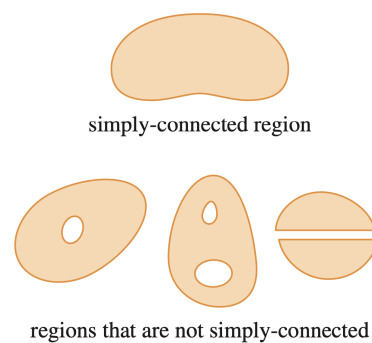
Theorem. Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D .



Definition. What is a simple curve?



Definition. What is a simply-connected region?

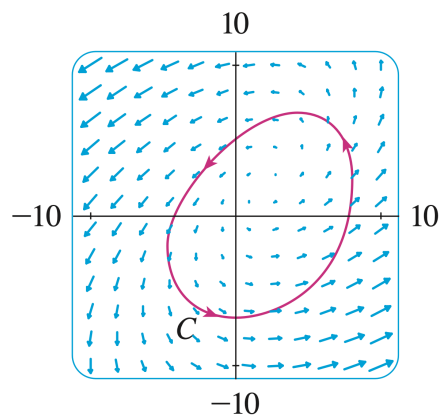


Theorem. How can we determine whether or not a vector field $\mathbf{F}$ is conservative?

Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$$

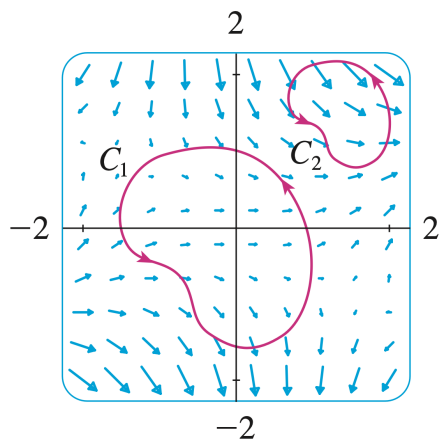
is conservative.



Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$$

is conservative.



Example.

(a) If $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t \mathbf{i} + e^t \cos t \mathbf{j}, \quad 0 \leq t \leq \pi$$