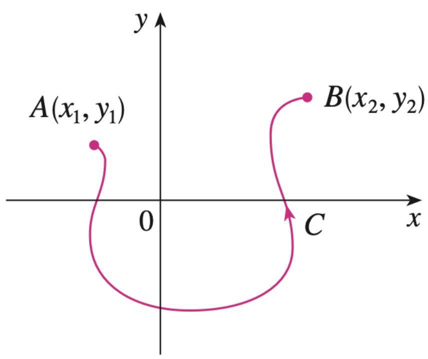
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

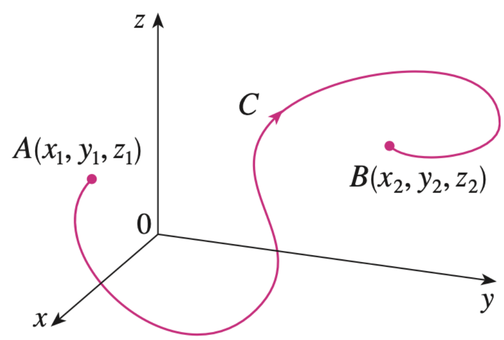
Conservative : $\vec{F} = \nabla f$

13.3 The Fundamental Theorem for Line Integrals

Theorem (Fundamental Theorem for Line Integrals)**.** Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. What is $\int_C \nabla f \cdot d\mathbf{r}$?



(a)



(b)

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(x_2, y_2) - f(x_1, y_1)$$

Note: This says we can evaluate the line integral of a conservative vector field just by knowing the value of f at the endpoints.

Proof.

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}(x(t)), \frac{\partial f}{\partial y}(y(t)), \frac{\partial f}{\partial z}(z(t)) \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

Chain rule

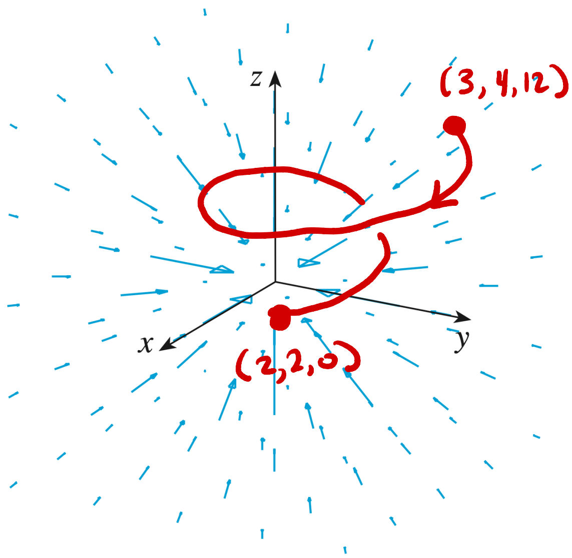
$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt$$

F.T.C.

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

□

Example. Find the work done by gravity in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a smooth curve C.



- The magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

- Place the object with mass M at the origin of $\mathbb{R}^3$ and the object with mass m at $\mathbf{x} = \langle x, y, z\rangle$. The gravitational force acts toward the origin, in the direction of the unit vector $-\dfrac{\mathbf{x}}{|\mathbf{x}|}$.

- Hence, the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

- Since $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$, we can write $\mathbf{F}$ in terms of its component functions as

$$\mathbf{F}(x, y, z) = \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}$$

Note: $\vec{F}$ is a conservative vector field. Indeed,

$$\vec{F} = \nabla f, \quad \text{where } f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}.$$

f is called a potential function.

Therefore, the work done is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= f(2,2,0) - f(3,4,12)$$

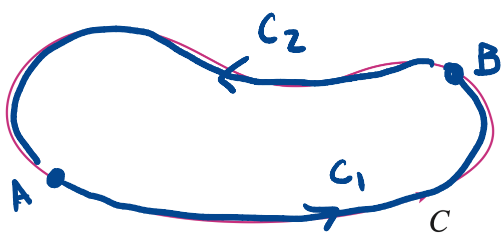
$$= \frac{mMG}{\sqrt{2^2+2^2}} - \frac{mMG}{\sqrt{3^2+4^2+12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)$$

Definition. If $\mathbf{F}$ is a continuous vector field with domain D, what does it mean for $\int_C \mathbf{F} \cdot d\mathbf{r}$ to be independent of path?

$\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D that have the same endpoints. Line integrals of conservative vector fields are independent of path. In general $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$.

Theorem. Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

$\oint_C$ means C is a closed path.

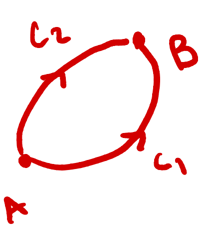


($\Rightarrow$) If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path, choose any two points A and B on C.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$

Since C_1 and $-C_2$ have the same initial and terminal points.

($\Leftarrow$) If $\int_C \vec{F} \cdot d\vec{r} = 0$ when C is a closed path, choose any two points A and B and any two paths C_1 and C_2 from A to B. Let C be the curve consisting of C_1 followed by $-C_2$.

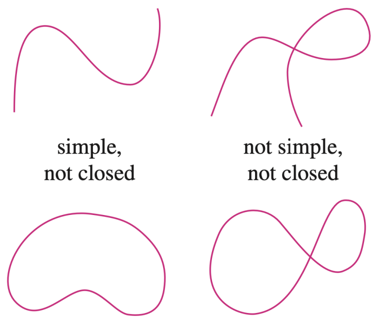


$$0 = \oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

<ins>Rmk</ins>: The line integral of any conservative vector field along a closed path is 0.

Thm: If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path on D, then $\vec{F}$ is a conservative vector field on D, i.e. $\nabla f = \vec{F}$ for some f.

Definition. What is a simple curve?

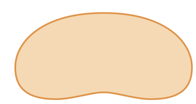


simple, not closed — not simple, not closed — simple, closed — not simple, closed

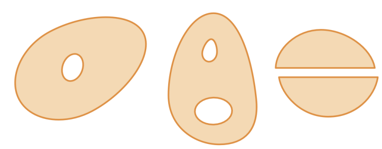
A simple curve is a curve that doesn't intersect itself

Definition. What is a simply-connected region?

- There are no holes
- There is a path between any two points



simply-connected region



regions that are not simply-connected

Theorem. How can we determine whether or not a vector field **F** is conservative?

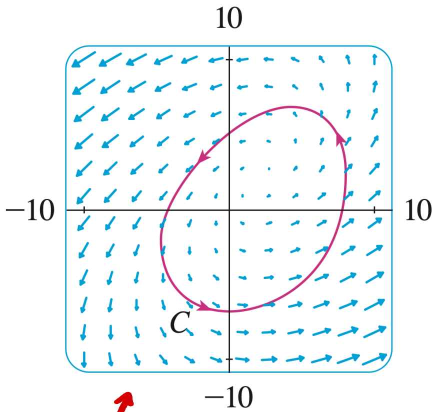
($\Rightarrow$)
- If $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative, then there is some function f such that $\vec{F} = \nabla f$.
- This means $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$
- By Clairaut's Theorem, $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$
- Conclude: If $\vec{F}$ is conservative, we must have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

($\Leftarrow$) Thm: If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and the region D is open and simply-connected, then $\vec{F}$ is conservative on D.

Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$$

is conservative.



- Let $P(x,y) = x - y$ and $Q(x,y) = x - 2$
- Then

$$\frac{\partial P}{\partial y} = -1 \quad \text{and} \quad \frac{\partial Q}{\partial x} = 1$$

- Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\vec{F}$ is not conservative

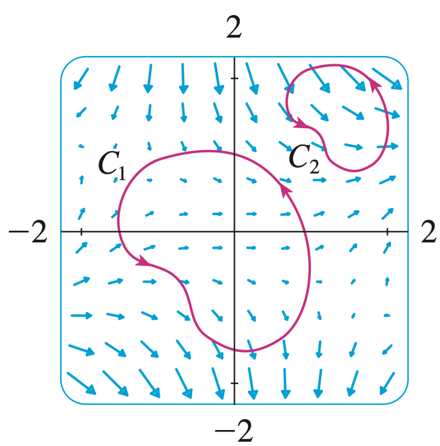
All the arrows point in roughly the same direction as the curve

$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} > 0$

Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

is conservative.



- Let $P(x,y) = 3 + 2xy$ and $Q(x,y) = x^2 - 3y^2$
- $\frac{\partial P}{\partial y} = 2x$ and $\frac{\partial Q}{\partial x} = 2x$
- Since the domain of $\vec{F}$ is $\mathbb{R}^2$, which is open and simply-connected, $\vec{F}$ is conservative.

Plausible that line integrals are 0 (things seem to cancel as we traverse a loop)

Example.

(a) If $\mathbf{F}(x,y) = (3+2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by

$$\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}, \qquad 0 \le t \le \pi$$

(a) Since $\vec{F}$ is conservative (see previous example), there is some f with $\nabla f = \vec{F}$. This means

$$\underbrace{f_x(x,y) = 3 + 2xy}_{\text{EQ1}} \qquad \underbrace{f_y(x,y) = x^2 - 3y^2}_{\text{EQ2}}$$

Integrate EQ1 with respect to x: $\underbrace{f(x,y) = 3x + x^2y + g(y)}_{\text{EQ3}}$

Differentiate EQ3 with respect to y: $f_y(x,y) = x^2 + g'(y)$

Conclude: $g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + K$

Plugging this back into EQ3, $\boxed{f(x,y) = 3x + x^2y - y^3 + K}$

(b) • We just need to know the initial point and the terminal point since $\vec{F}$ is conservative, and we know the potential function is $f(x,y) = 3x + x^2y - y^3$ (choose $K=0$).

• $\vec{r}(0) = (0,1)$ and $\vec{r}(\pi) = (0, -e^{\pi})$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(0,-e^{\pi}) - f(0,1) = e^{3\pi} - (-1) = \boxed{e^{3\pi}+1}$$