Lecture Notes
Math 2400 - Calculus III
Spring 2024

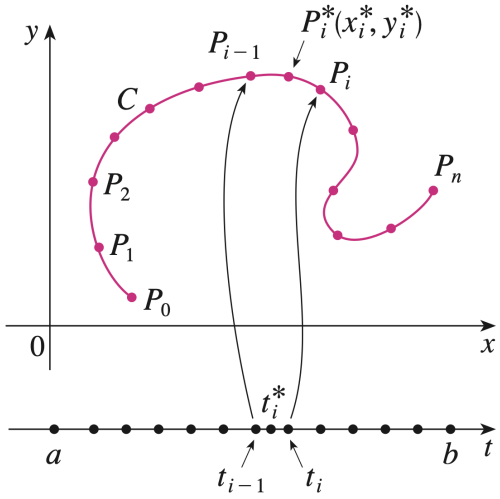
Name: _____

13.2 Line Integrals

Definition. If f is defined on a smooth curve C given by

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

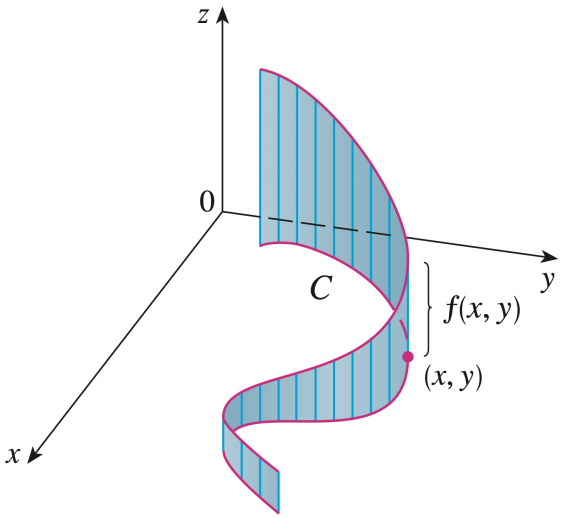
what is the line integral of f along C?



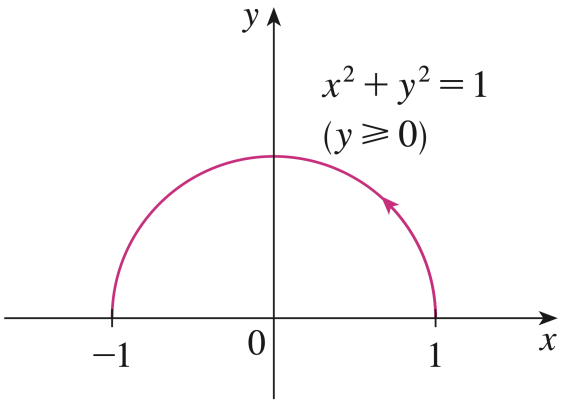
Definition. Suppose that a smooth curve C is defined parametrically by the equations

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b$$

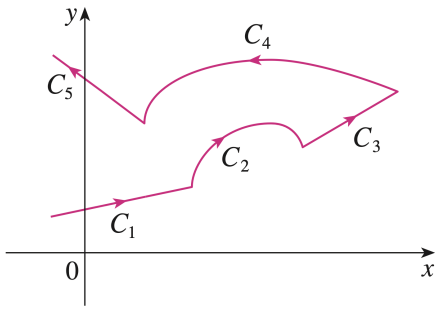
If f is a continuous function, how can we evaluate the line integral of f along the curve C?



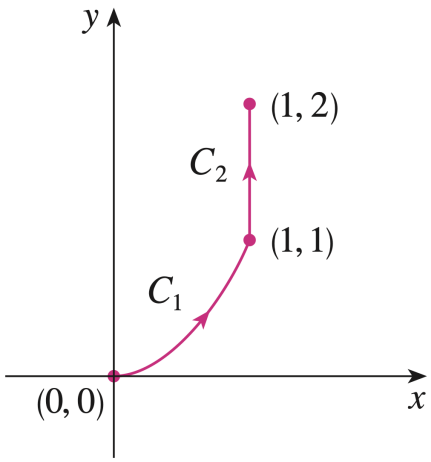
Example. Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.



Question. How can we define the line integral of f along C if C is a piecewise-smooth curve?



Example. Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

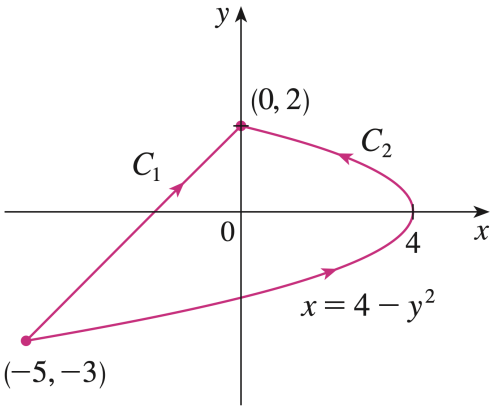


Definition. What are the line integrals of f along C with respect to x and y?

Example. Evaluate $\int_C y^2\,dx + x\,dy$, where

(a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$.

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.



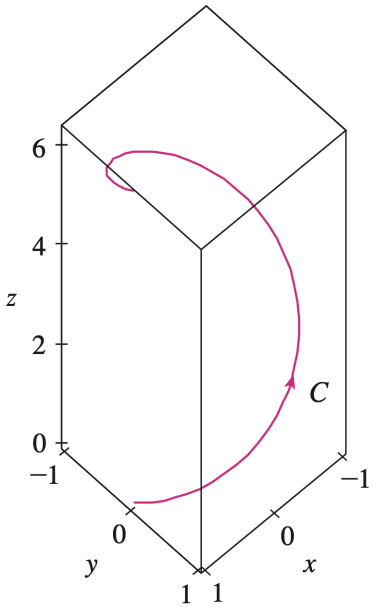
Definition. Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \leq t \leq b$$

If f is a function of three variables that is continuous on some region containing C, what is the line integral of f along C?

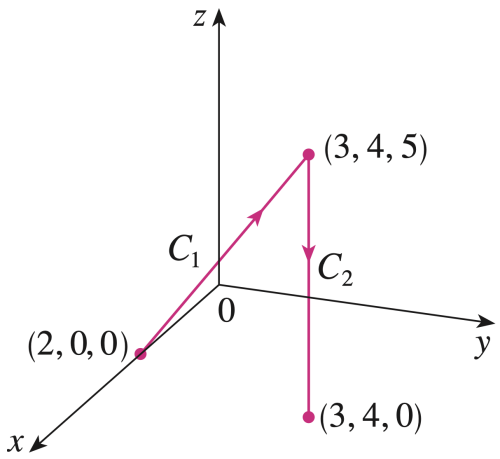
Remark. How can we rewrite $\int_C f(x, y)\, ds$ and $\int_C f(x, y, z)\, ds$ more compactly?

Example. Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t, y = \sin t, z = t$ for $0 \leq t \leq 2\pi$.

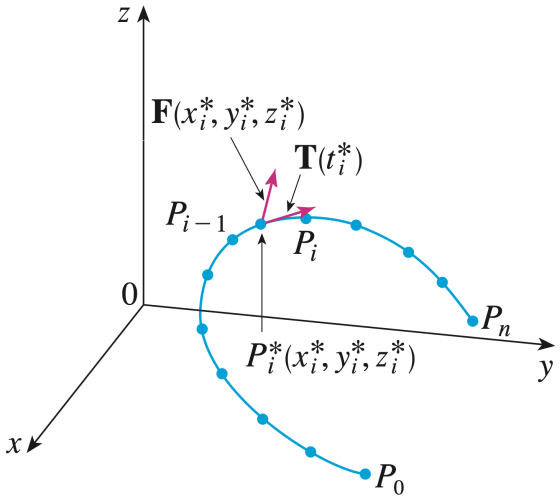


Question. If C is a space curve, line integrals along C with respect to x, y, and z can also be defined. For example, what is $\int_C f(x, y, z) \, dz$?

Example. Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$, followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

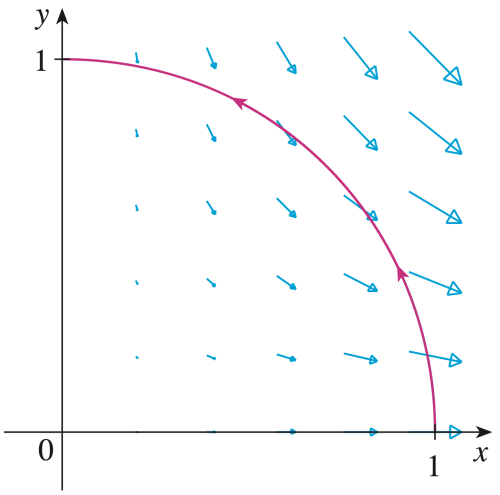


Question. Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. How can we compute the work done by this force in moving a particle along a smooth curve C?



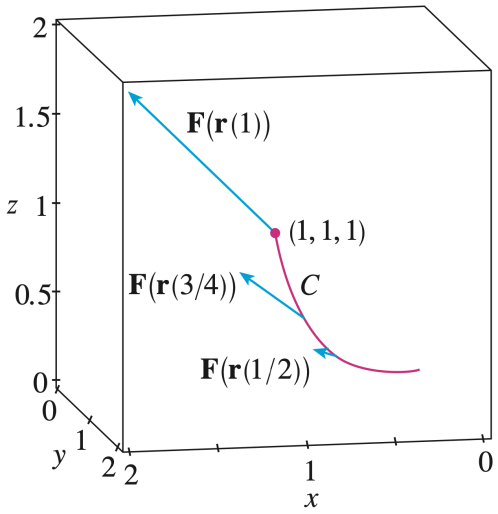
Definition. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$ for $a \leq t \leq b$. What is the line integral of $\mathbf{F}$ along C?

Example. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \leq t \leq \pi/2$.



Example. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \le t \le 1$$



Question. What is the relationship between line integrals of vector fields and line integrals of scalar fields?