Lecture Notes
Math 2400 - Calculus III
Spring 2024

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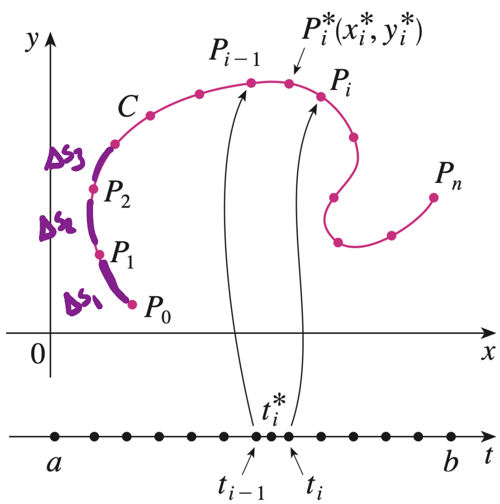
More accurate to call these "curve integrals"

13.2 Line Integrals

Definition. If f is defined on a smooth curve C given by

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

what is the line integral of f along C?



- Divide $[a,b]$ into n subintervals $[t_{i-1}, t_i]$
- This divides C into n subarcs with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$
- Choose a point $P_i^*(x_i^*, y_i^*)$ in the ith subarc

height base

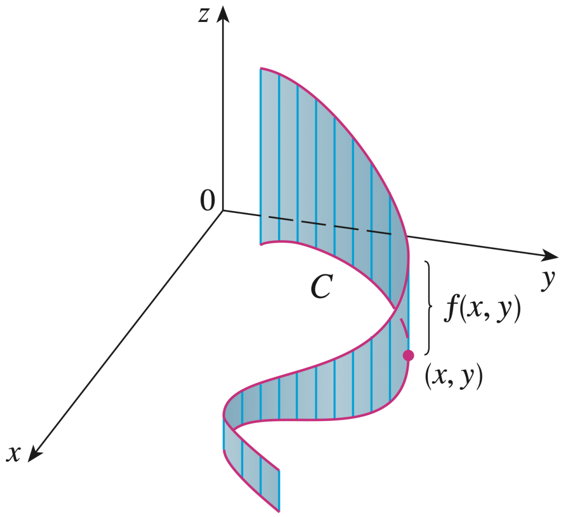
- The sum $\sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta s_i$ approximates the area below $f(x,y)$ and above the curve C
- The line integral of f along the curve C is

$$\int_C f(x,y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

Definition. Suppose that a smooth curve C is defined parametrically by the equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

If f is a continuous function, how can we evaluate the line integral of f along the curve C?



if $f(x,y) \ge 0$, $\int_C f(x,y)\,ds$ represents the area of the "curtain" under f and above C.

In terms of t, the length ds is

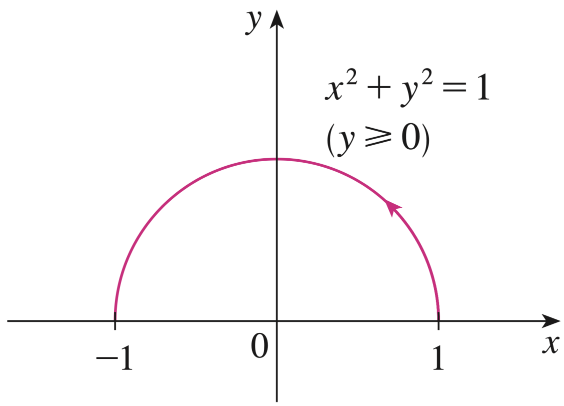
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

See §10.3

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt}$$

- This is called the line integral of f with respect to arclength.
- This integral does not depend on the parametrization of the curve.

Example. Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.



① Find a parametrization for C.

$$\Rightarrow \quad \vec{r}(t) = \langle \overset{x(t)}{\overset{\shortparallel}{\cos t}},\ \overset{y(t)}{\overset{\shortparallel}{\sin t}} \rangle \quad \text{where} \quad 0 \leq t \leq \pi$$

② Here, $f(x,y) = 2 + x^2 y$. So $f(x(t), y(t)) = 2 + \cos^2 t \cdot \sin t$

Also, $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t$

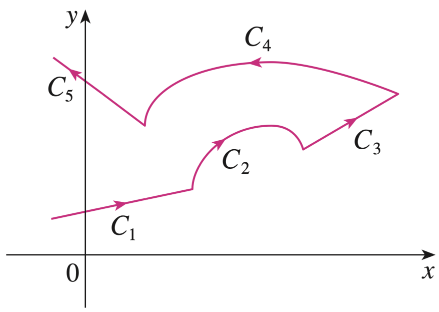
③ $$\int_C 2 + x^2 y \, ds = \int_0^\pi (2 + \cos^2 t \cdot \sin t) \cdot \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^\pi 2 + \cos^2 t \cdot \sin t \, dt$$

$$= \left[2t - \frac{\cos^3 t}{3} \right]_{t=0}^{t=\pi}$$

$$= 2\pi + \frac{2}{3}$$

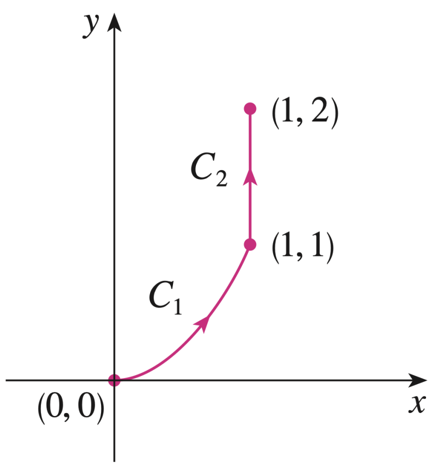
Question. How can we define the line integral of f along C if C is a piecewise-smooth curve?



We can integrate each piece separately and take the sum:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds$$

Example. Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.



1. A parametrization for C_1 is

$$\vec{r}_1(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1$$

$$\int_{C_1} 2x\,ds = \int_0^1 2t \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^1 2t \cdot \sqrt{1 + 4t^2}\,dt$$

$$= \left[\frac{1}{4} \cdot \frac{2}{3} (1 + 4t^2)^{3/2}\right]_{t=0}^{t=1} = \frac{5\sqrt{5} - 1}{6}$$

2. A parametrization for C_2 is

$$\vec{r}_2(t) = \langle 1, t \rangle, \quad 1 \le t \le 2$$

$$\int_{C_2} 2x\,ds = \int_1^2 2 \cdot 1 \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^2 2\,dt = 2$$

3. Hence $\int_C 2x\,ds = \dfrac{5\sqrt{5} - 1}{6} + 2$

Definition. What are the line integrals of f along C with respect to x and y?

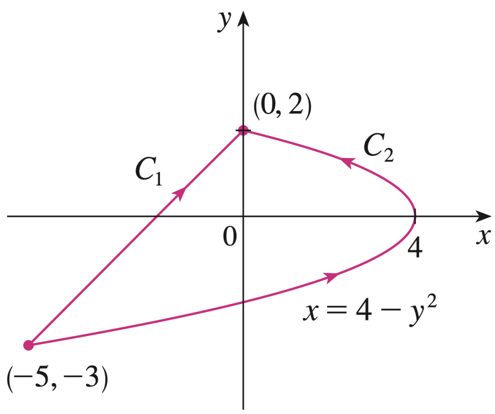
$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t)) \cdot x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t)) \cdot y'(t)\,dt$$

Example. Evaluate $\int_C y^2\,dx + x\,dy$, where $\longrightarrow \int_C y^2\,dx + \int_C x\,dy$

(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$.

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.



(a) A parametrization for C_1 is

$$\vec{c}_1(t) = \langle 5t-5, 5t-3 \rangle, \quad 0 \le t \le 1$$

$\left[\text{Recall: } \vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, \ 0 \le t \le 1\right]$

Since $dx = 5\,dt$ and $dy = 5\,dt$,

$$\int_{C_1} y^2\,dx + x\,dy = \int_0^1 (5t-3)^2 \cdot (5\,dt) + (5t-5)\cdot(5\,dt)$$

$$= 5\int_0^1 25t^2 - 25t + 4\,dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_{t=0}^{t=1}$$

$$= \frac{-5}{6}$$

(b) A parametrization for C_2 is

$$\vec{c}_2(t) = \langle 4-t^2, t \rangle \quad \text{for} \quad -3 \leq t \leq 2$$

Since $dx = -2t\,dt$ and $dy = dt$,

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} t^2(-2t\,dt) + (4-t^2)\,dt$$

$$= \int_{-3}^{2} -2t^3 - t^2 + 4\,dt$$

$$= \left[-\frac{t^4}{2} - \frac{t^3}{3} + 4t \right]_{t=-3}^{t=2}$$

$$= 40\tfrac{5}{6}$$

Note: We got different answers, even though the curves had the same endpoints

Big Question: When is the line integral independent of the path? (* suspenseful music *)

Definition. Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

If f is a function of three variables that is continuous on some region containing C, what is the line integral of f along C?

- This is similar to line integrals for plane curves

- $\displaystyle\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$

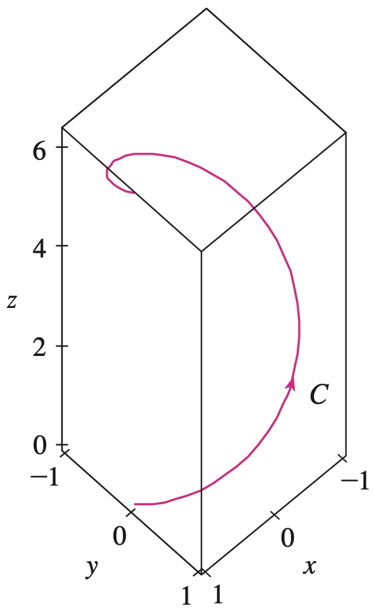
$$= \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Remark. How can we rewrite $\int_C f(x,y)\,ds$ and $\int_C f(x,y,z)\,ds$ more compactly?

- If C is parametrized by $\vec{r}(t)$, then both integrals can be written as $\displaystyle\int_a^b f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|\, dt$

- Note: $f(x,y,z) = 1$, then $\displaystyle\int_C ds = \int_a^b |\vec{r}\,'(t)|\, dt$ is the arclength of C (See §10.3)

Example. Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t, y = \sin t, z = t$ for $0 \le t \le 2\pi$.



$$\int_C y \sin z \, ds$$

$$\int_0^{2\pi} (\sin t) \sin t \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$\int_0^{2\pi} \sin^2 t \cdot \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} \, dt$$

$$\sqrt{2} \int_0^{2\pi} \frac{1 - \cos 2t}{2} \, dt$$

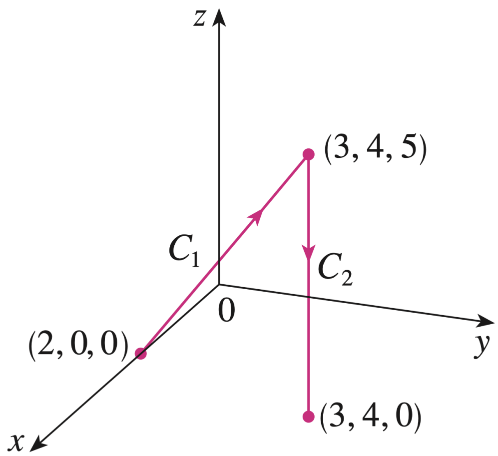
$$\frac{\sqrt{2}}{2} \left[t - \frac{1}{2} \sin 2t \right]_{t=0}^{t=2\pi}$$

$$\sqrt{2} \cdot \pi$$

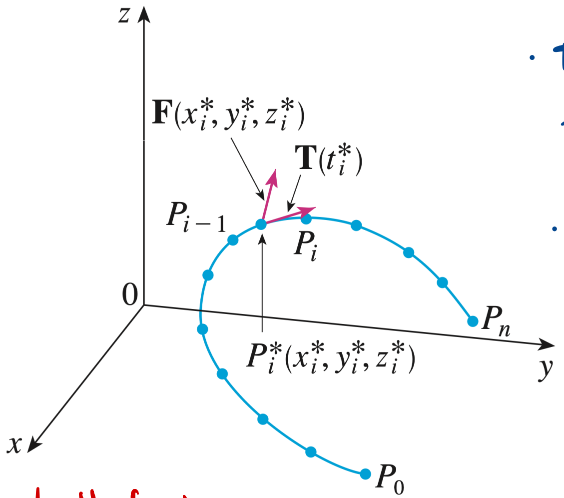
Question. If C is a space curve, line integrals along C with respect to x, y, and z can also be defined. For example, what is $\int_C f(x, y, z) \, dz$?

$$\int_C f(x,y,z) \, dz = \int_a^b f(x(t), y(t), z(t)) \cdot z'(t) \, dt$$

Example. Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$, followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$.



Question. Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a continuous force field on $\mathbb{R}^3$. How can we compute the work done by this force in moving a particle along a smooth curve C?



- The work done by a constant force $\vec{F}$ is $\vec{F} \cdot \vec{d}$
- To compute the work done by a variable force (i.e. a continuous vector field), we use line integrals.

length of subarc

parameter corresponding to the sample point P_i^*

- If Δs_i is small enough, the particle proceeds approximately in the direction of the unit tangent vector $\vec{T}(t_i^*)$
- The work done by $\vec{F}$ in moving the particle from P_{i-1} to P_i is approximately

$$\overset{\vec{F}}{\vec{F}(x_i^*, y_i^*, z_i^*)} \cdot \overset{\vec{d}}{\left[\Delta s_i \vec{T}(t_i^*)\right]} = \left[\vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(t_i^*)\right] \Delta s_i$$

- The total work done is approximately

$$\sum_{i=1}^{n} \left[\vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(x_i^*, y_i^*, z_i^*)\right] \Delta s_i$$

- As n becomes large, we obtain

$$W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\, ds = \int_C \vec{F} \cdot \vec{T}\, ds$$

$\vec{F}(\vec{r}(t))$ means $\vec{F}(x(t), y(t), z(t))$

$d\vec{r}$ means $\vec{r}\,'(t)\,dt$

Definition. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$ for $a \le t \le b$. What is the line integral of $\mathbf{F}$ along C?

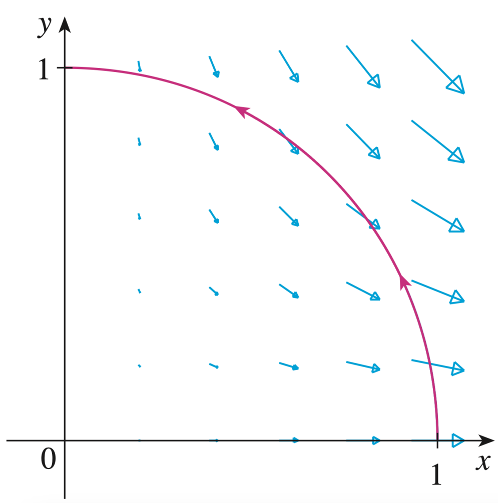
If C is given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, then

$\vec{T}(t) = \vec{r}\,'(t) / |\vec{r}\,'(t)|$ and $ds = |\vec{r}\,'(t)|\,dt$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \cdot |\vec{r}\,'(t)|\,dt = \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}$$

often written as $\int_C \vec{F}\cdot d\vec{r}$

Example. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ for $0 \le t \le \pi/2$.



① $\vec{r}(t) = \langle \overset{x(t)}{\cos t}, \overset{y(t)}{\sin t} \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\vec{F}(\vec{r}(t)) = \langle \cos^2 t, -\cos t \sin t \rangle$

② $$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

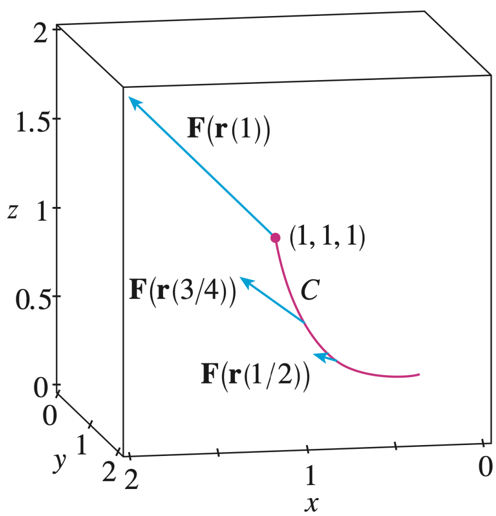
$$= \int_0^{\pi/2} \langle \cos^2 t, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\,dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t\,dt$$

$$= \left[\frac{2}{3}\cos^3 t \right]_{t=0}^{t=\pi/2} = -\frac{2}{3}$$

Example. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3 \qquad 0 \le t \le 1$$



① $\vec{r}(t) = \langle \underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3} \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t^3, t^5, t^4 \rangle$

② $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$= \int_0^1 \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\, dt$

$= \int_0^1 t^3 + 5t^6\, dt$

$= \left[\frac{t^4}{4} + \frac{5t^7}{7} \right]_{t=0}^{t=1}$

$= \frac{27}{28}$

Question. What is the relationship between line integrals of vector fields and line integrals of scalar fields?

Suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \langle P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_a^b P(x(t), y(t), z(t))\, x'(t)\, dt + \int_a^b Q(x(t), y(t), z(t))\, y'(t)\, dt + \int_a^b R(x(t), y(t), z(t))\, z'(t)\, dt$$

$$= \int_C P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz$$

Example: $\int_C y\, dx + z\, dy + x\, dz = \int_C \vec{F}\cdot d\vec{r}$

where $\vec{F}(x,y,z) = \langle y, z, x \rangle$