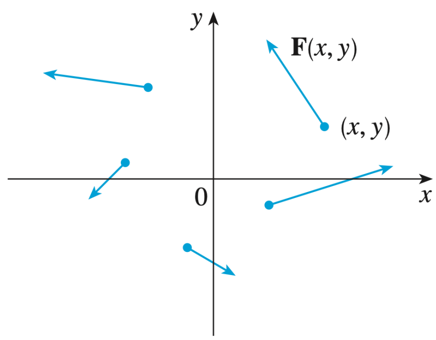
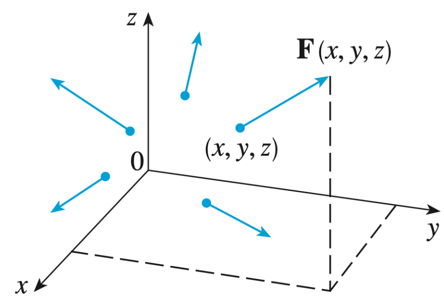
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

13.1 Vector Fields

Definition. What is a vector field?





A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) a two-dimensional vector $\vec{F}(x,y)$.

$$\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$$

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

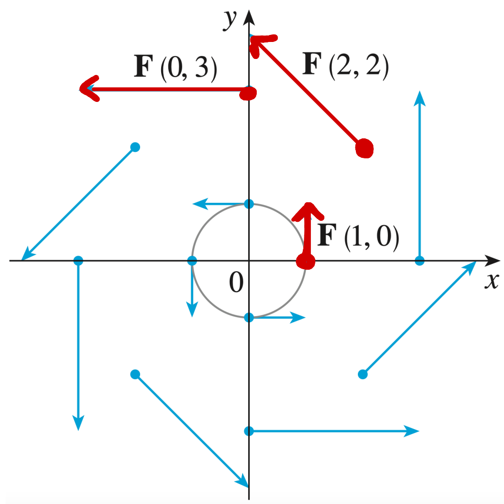
$$\vec{F}(x,y) = P\vec{i} + Q\vec{j}$$

P and Q are the component functions.

A vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point (x,y,z) a three-dimensional vector $\vec{F}(x,y,z)$

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

Example. A vector field on $\mathbb{R}^2$ is defined by $\vec{F}(x,y) = \langle -y, x\rangle$. Describe $\vec{F}$ by sketching some of the vectors $\vec{F}(x,y)$.



(x,y)	$\vec{F}(x,y)$
$(1,0)$	$\langle 0,1\rangle$
$(2,2)$	$\langle -2,2\rangle$
$(0,3)$	$\langle -3,0\rangle$

Note: Each arrow is tangent to a circle centered at the origin.
Given a point (x,y) with position vector $\langle x,y\rangle$

$$\langle x,y\rangle \cdot \vec{F}(x,y) = \langle x,y\rangle \cdot \langle -y,x\rangle = -yx+yx=0$$

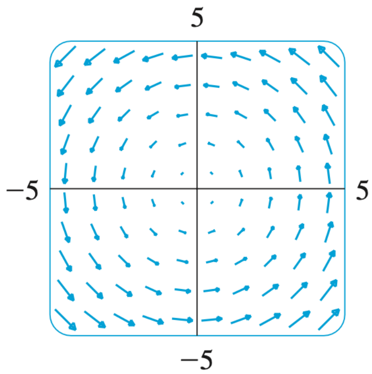
So $\vec{F}(x,y)$ is perpendicular to the position vector $\langle x,y\rangle$

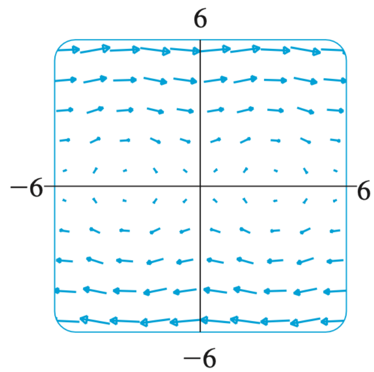
Note: The radius of this circle is $|\langle x,y\rangle| = \sqrt{x^2+y^2}$

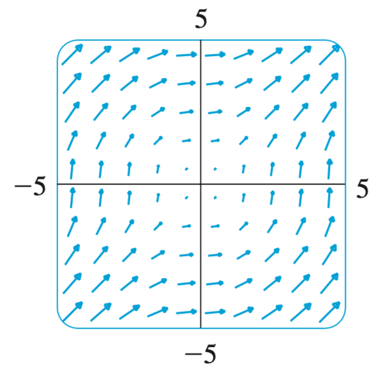
This is also the magnitude $|\vec{F}(x,y)| = \sqrt{(-y)^2+x^2}$

* Be able to match vector fields to their equations

Example. Examine the vector fields below.







$\vec{F}(x,y) = \langle -y, x \rangle$

Good example to know

$\vec{F}(x,y) = \langle y, \sin x \rangle$

As y increases, the $\vec{i}$-component increases

$\vec{F}(x,y) = \langle \ln(1+y^2), \ln(1+x^2) \rangle$

If x and y are large enough, both $\vec{i}$ and $\vec{j}$ components are positive

Definition. What is a gradient vector field? What is a conservative vector field?

If $f(x,y)$ is a scalar function, its gradient

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

defines a vector field called the gradient vector field.

A vector field $\vec{F}(x,y)$ is called conservative if it is the gradient of some scalar function. If $\vec{F} = \nabla f$, then f is called the potential function for $\vec{F}$.