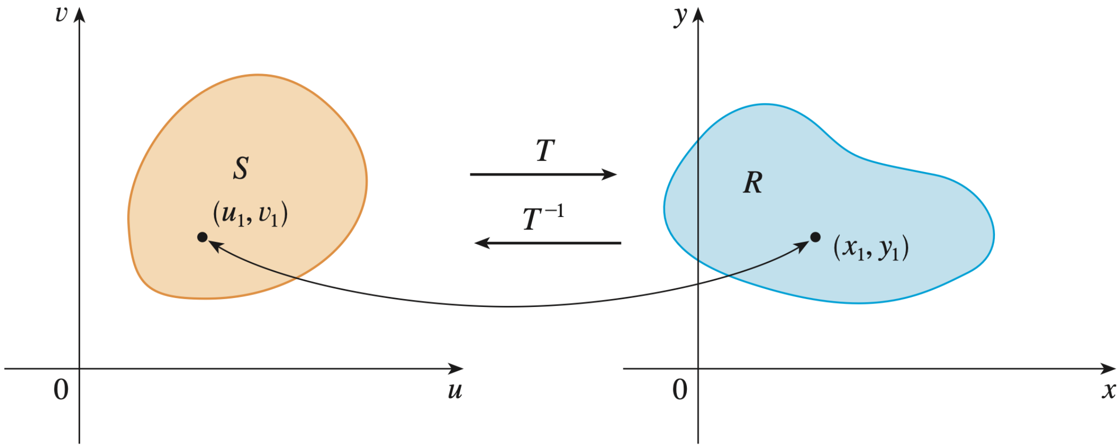
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: ____________________________

12.9 Change of Variables in Multiple Integrals

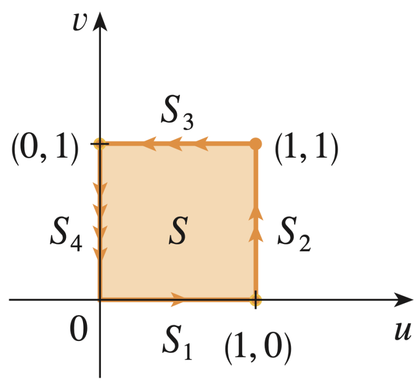
Definition. What is a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$?

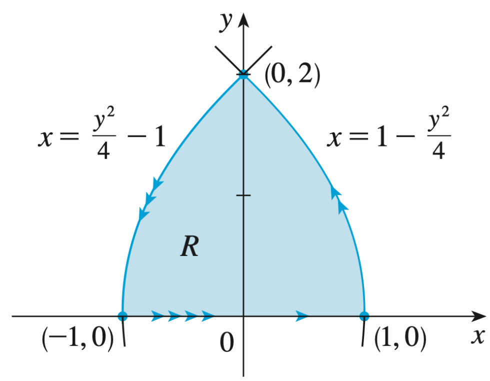


Example. A transformation is defined by the equations

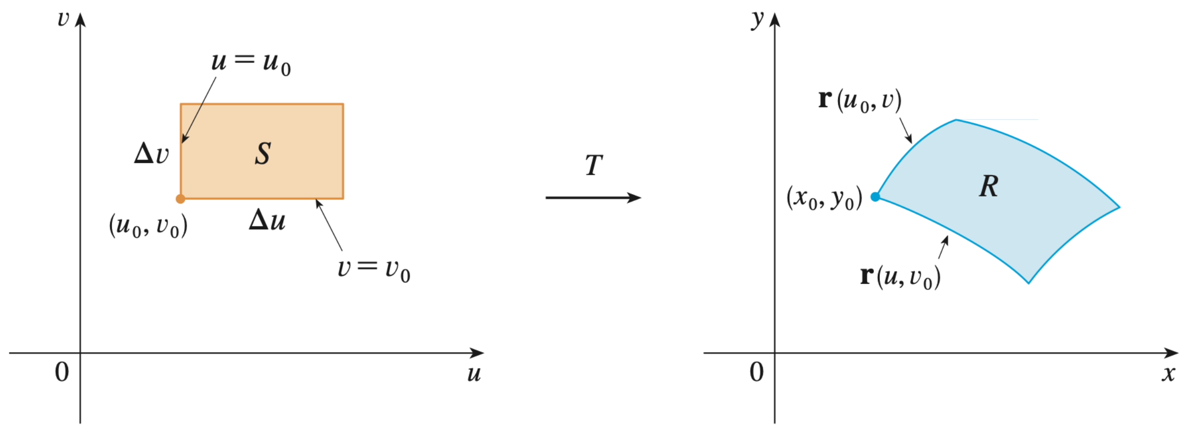
$$x = u^2 - v^2, \qquad y = 2uv$$

Find the image of the square $S = [0, 1] \times [0, 1]$.





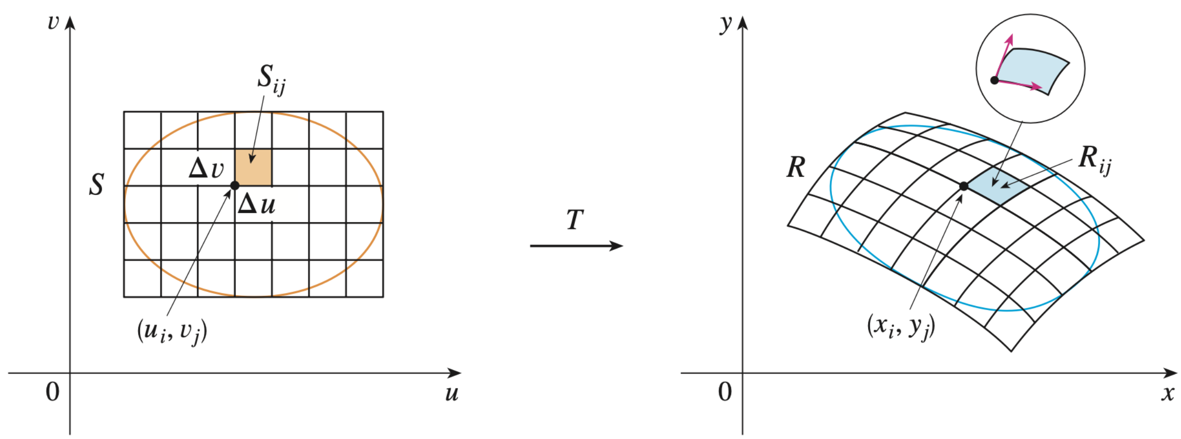
Question. Suppose that a rectangle S in the uv-plane is mapped to a region R in the xy-plane under a transformation T. If S has dimensions Δu and Δv, how can we approximate the area of the region R?



Definition. What is the Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$?

Remark. Use the Jacobian to give an approximation to the area ΔA of the region R above.

Theorem. How can we compute the double integral of f over R using a general change of coordinates?

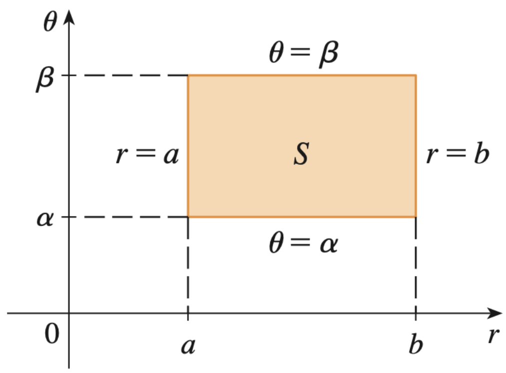


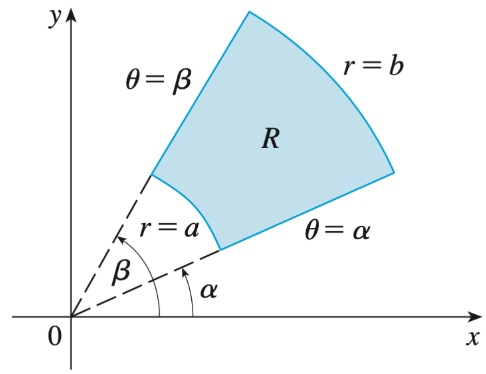
$$\iint\limits_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n}$$

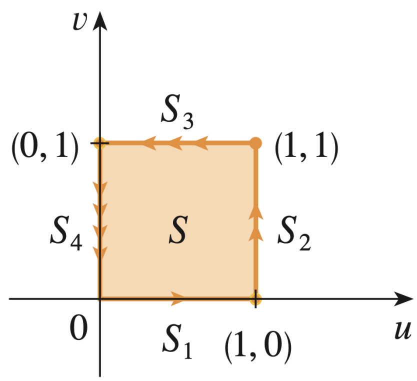
$$\approx \iint\limits_S f\Big(g(u, v), h(u, v)\Big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

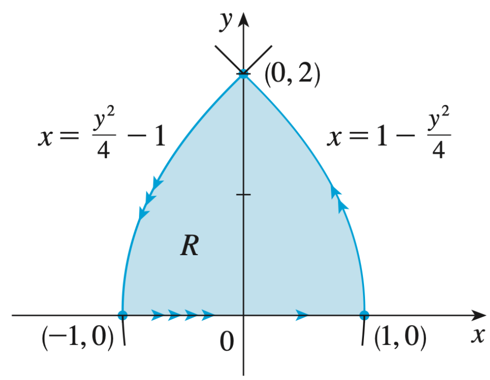
Example. Show that integration in polar coordinates is just a special case of the change of coordinates formula above.



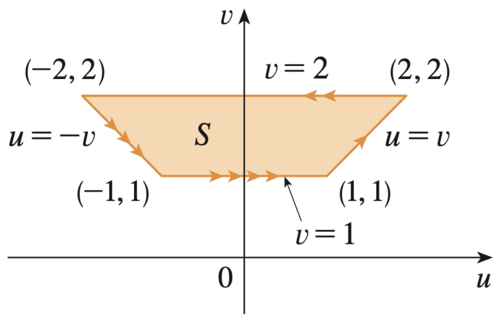


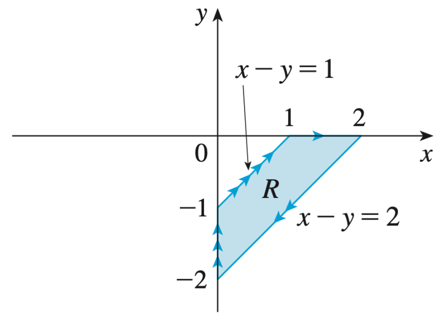
Example. Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.





Example. Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.





Theorem. What is the change of variables formula for triple integrals?

Example. Derive the formula for triple integration in spherical coordinates.