Lecture Notes
Math 2400 - Calculus III
Spring 2024

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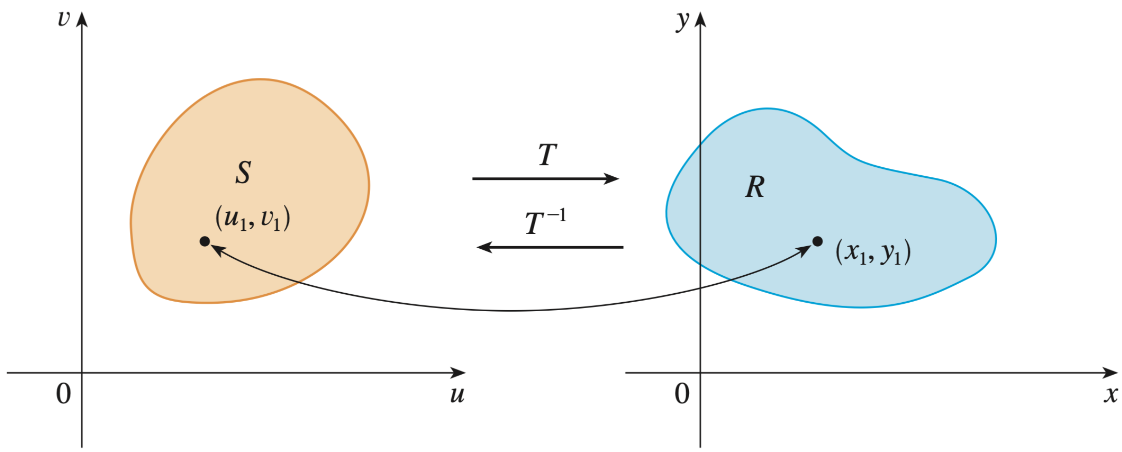
12.9 Change of Variables in Multiple Integrals

Definition. What is a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$?

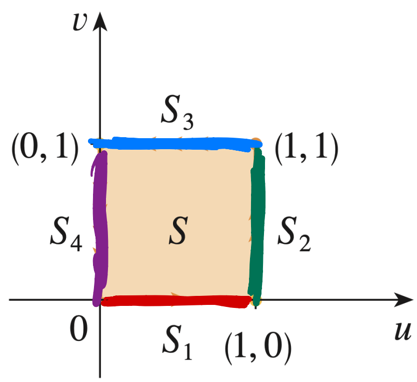


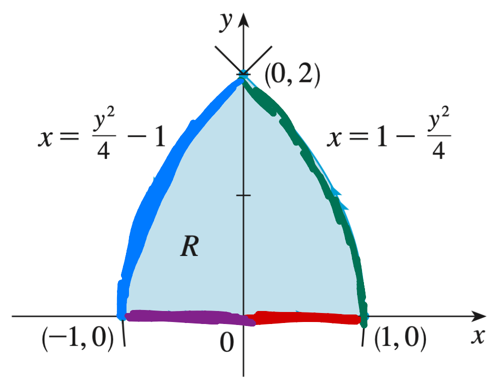
- A transformation T is a function whose domain and range are both subsets of $\mathbb{R}^2$.
- $T(u,v) = (x,y)$ where $x = g(u,v)$ and $y = h(u,v)$
- If no two points have the same image, T is called <u>one-to-one</u> (or injective). If T is one-to-one, then it has an inverse T^{-1}.
- We will usually assume that T is a C^1 transformation. This means that g and h have continuous first order partial derivatives.

Example. A transformation is defined by the equations

$$x = u^2 - v^2, \qquad y = 2uv$$

Find the image of the square $S = [0,1] \times [0,1]$.





- The side S_1 is given by $v = 0$ $(0 \leq u \leq 1)$

  $\Rightarrow$ $x = u^2$ and $y = 0$

  $\Rightarrow$ S_1 is mapped to the line segment from $(0,0)$ to $(1,0)$

- The side S_2 is given by $u = 1$ $(0 \leq v \leq 1)$

  $\Rightarrow$ $x = 1 - v^2$ and $y = 2v$

  $\Rightarrow$ Eliminating v, $x = 1 - \frac{y^2}{4}$ for $0 \leq y \leq 2$

- The side S_3 is given by $v = 1$ $(0 \leq u \leq 1)$

  $\Rightarrow$ $x = u^2 - 1$ and $y = 2u$

  $\Rightarrow$ Eliminating u, $x = \frac{y^2}{4} - 1$ for $0 \leq y \leq 2$

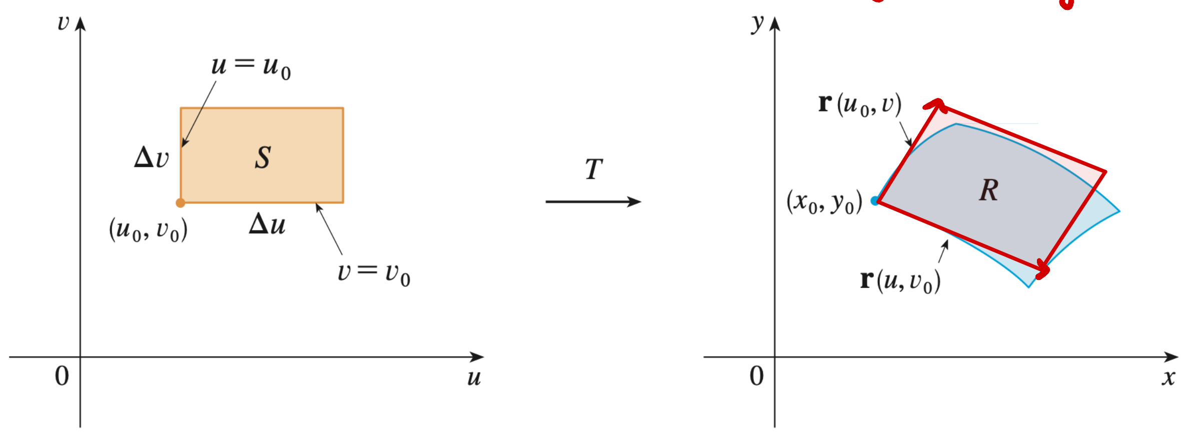
- The side S_4 is given by $u = 0$ $(0 \leq v \leq 1)$

  $\Rightarrow$ $x = -v^2$ and $y = 0$

  $\Rightarrow$ S_4 is mapped to the line segment from $(-1,0)$ to $(0,0)$

Question. Suppose that a rectangle S in the uv-plane is mapped to a region R in the xy-plane under a transformation T. If S has dimensions Δu and Δv, how can we approximate the area of the region R?

Let $T(u,v) = (x,y)$, where $x = g(u,v)$ and $y = h(u,v)$



- Idea: Approximate R by a parallelogram. What are the sides?
- Given a point (u,v), $\vec{r}(u,v) = \langle g(u,v), h(u,v) \rangle$ is the position vector of the image of (u,v) under T.
- At (x_0, y_0), the tangent vectors are

$$\vec{r}_u = \langle g_u(u_0,v_0), h_u(u_0,v_0) \rangle = \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0) \right\rangle$$

$$\vec{r}_v = \langle g_v(u_0,v_0), h_v(u_0,v_0) \rangle = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0) \right\rangle$$

- The sides of the parallelogram are $\Delta u \, \vec{r}_u$ and $\Delta v \, \vec{r}_v$ ← See book for details
- The area of R is approximately

$$\left| (\Delta u \, \vec{r}_u) \times (\Delta v \, \vec{r}_v) \right| = |\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v$$

The Jacobian

where $$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k}$$

How much to scale the area of S to get the area of R

Definition. What is the Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$?

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

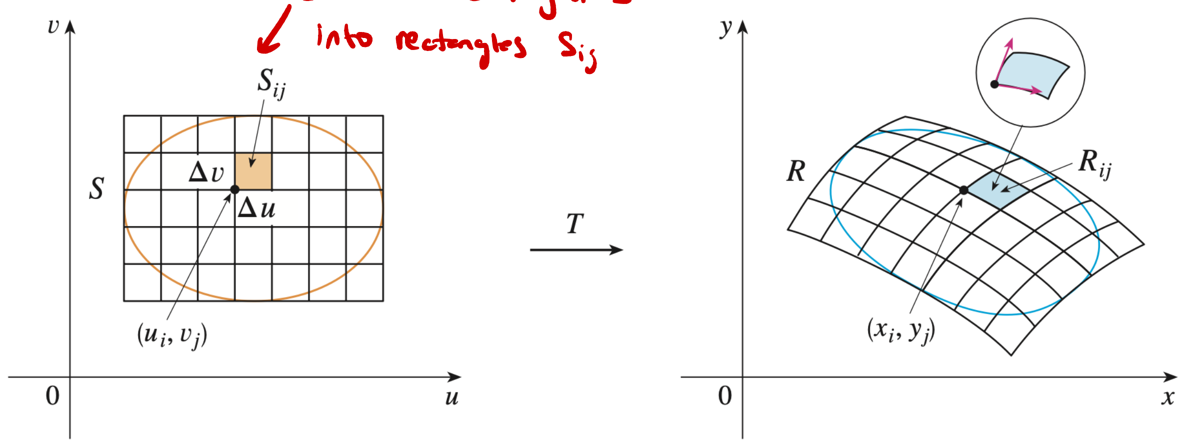
Remark. Use the Jacobian to give an approximation to the area ΔA of the region R above.

absolute value

$$\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$$

Theorem. How can we compute the double integral of f over R using a general change of coordinates?

Divide the region S into rectangles S_{ij}

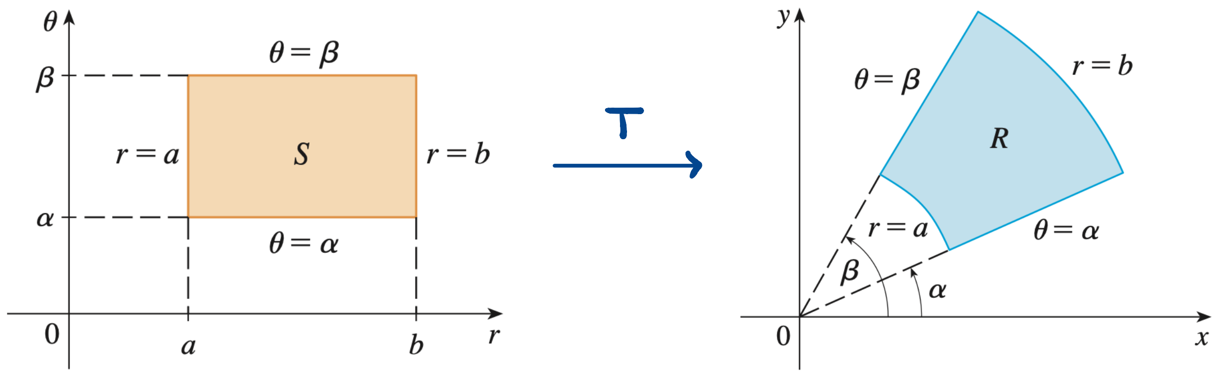


$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i,v_j), h(u_i,v_j)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

$$\approx \iint_S f\Big(g(u,v), h(u,v)\Big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Example. Show that integration in polar coordinates is just a special case of the change of coordinates formula above.



The transformation T goes from the $r\theta$-plane to the xy-plane

$$x = g(r,\theta) = r\cos\theta \qquad\qquad y = h(r,\theta) = r\sin\theta$$

The Jacobian is

Note: $r \geq 0$

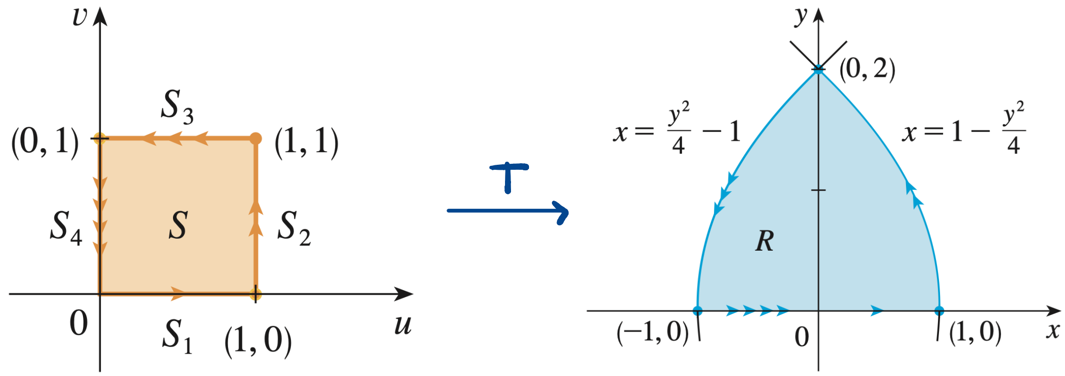
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

So...

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta$$

$$= \iint_S f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

Example. Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.



- This is the transformation from the example on pg. 2
- The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

Note: this is ≥ 0

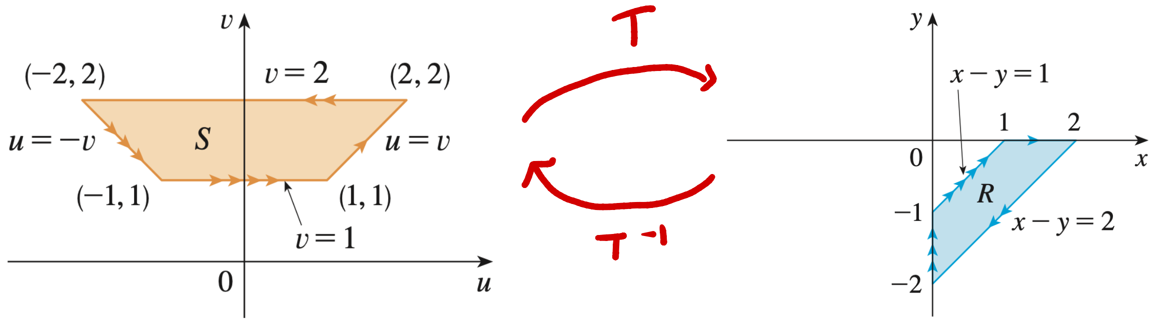
- $$\iint_R y\,dA = \iint_S 2uv \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA = \int_0^1 \int_0^1 2uv \cdot 4(u^2+v^2)\,du\,dv$$

$$= 8 \int_0^1 \int_0^1 u^3 v + uv^3 \,du\,dv$$

$$= 8 \int_0^1 \left[\tfrac{1}{4}u^4 v + \tfrac{1}{2}u^2 v^3 \right]_{u=0}^{u=1} dv$$

$$= \int_0^1 2v + 4v^3\,dv = \left[v^2 + v^4 \right]_{v=0}^{v=1} = 2$$

Example. Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\, dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.



- What should the change of variables be?

$$e^{\frac{x+y}{x-y}} \qquad x+y \to u, \quad x-y \to v$$

$$u = x+y \quad \text{and} \quad v = x-y \qquad \leftarrow T^{-1}$$

$$\Rightarrow u+v = 2x \quad \text{and} \quad u-v = 2y$$

$$\Rightarrow x = \tfrac{1}{2}(u+v) \quad \text{and} \quad y = \tfrac{1}{2}(u-v) \qquad \leftarrow T$$

- Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

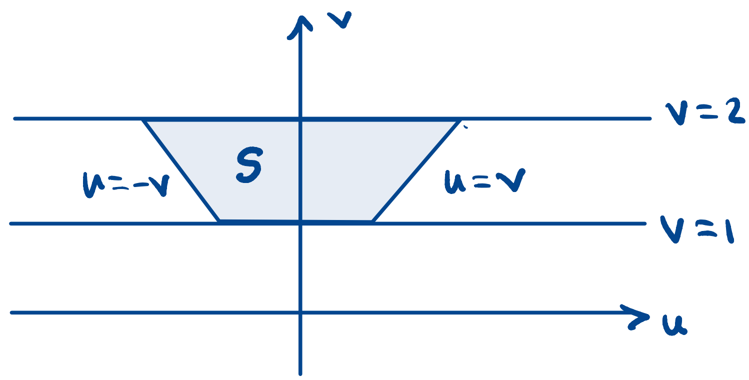
This is negative because going counterclockwise around S corresponds to going clockwise around R, so the orientation changes.

What is S? The sides of R are

$y=0 \qquad x-y=2 \qquad x=0 \qquad x-y=1$

In the uv-plane, these lines correspond to

$u=v \qquad v=2 \qquad u=-v \qquad v=1$



$\Rightarrow$ S is the trapezoidal region with vertices $(1,1)$, $(2,2)$, $(-2,2)$, and $(-1,1)$

$$\iint_R e^{\frac{x+y}{x-y}}\, dA = \iint_S e^{u/v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v} \cdot \frac{1}{2}\, du\, dv = \frac{1}{2}\int_1^2 \left[v e^{u/v} \right]_{u=-v}^{u=v} dv$$

$$= \frac{1}{2}\int_1^2 (e - e^{-1}) v \, dv$$

$$= \frac{3}{4}(e - e^{-1})$$

Theorem. What is the change of variables formula for triple integrals?

Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by

$$x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$$

The Jacobian of T is the following 3×3 determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

We then have

$$\iiint_R f(x,y,z)\, dV = \iiint_S f\big(g(u,v,w), h(u,v,w), k(u,v,w)\big) \cdot \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Example. Derive the formula for triple integration in spherical coordinates.

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

Expand along the bottom row

$$= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right)$$
$$- \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

use the fact that $\cos^2\theta + \sin^2\theta = 1$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin\phi\sin^2\phi = -\rho^2\sin\phi$$

Since $0 \le \phi \le \pi$, we have $\sin\phi \ge 0$, so $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \boxed{\rho^2\sin\phi}$