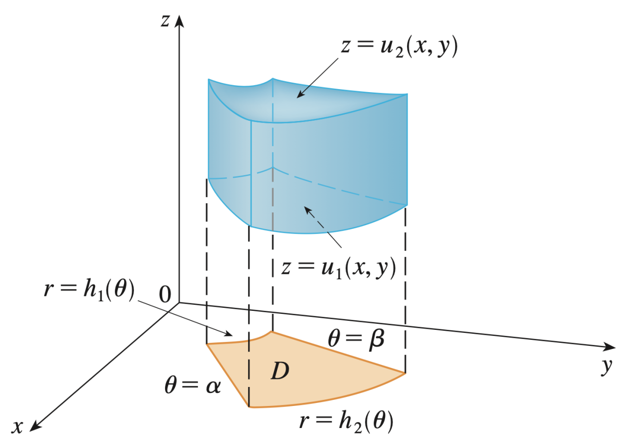
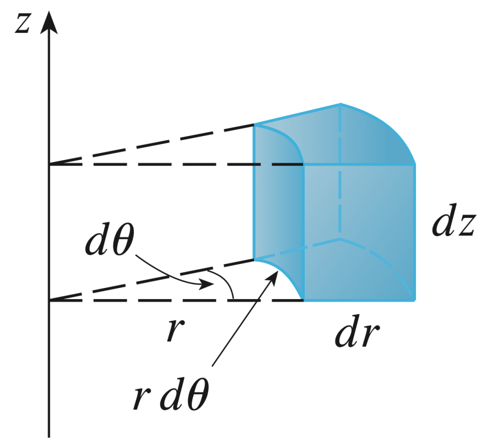
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: ____________________________

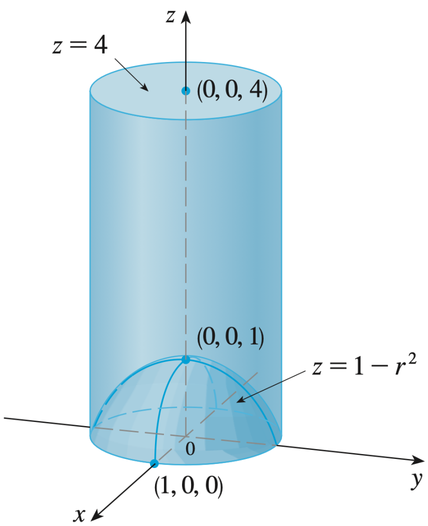
12.8 Triple Integrals in Cylindrical and Spherical Coordinates

Definition. Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates. How can we think about $\iiint_E f(x,y,z)\,dV$ using cylindrical coordinates?

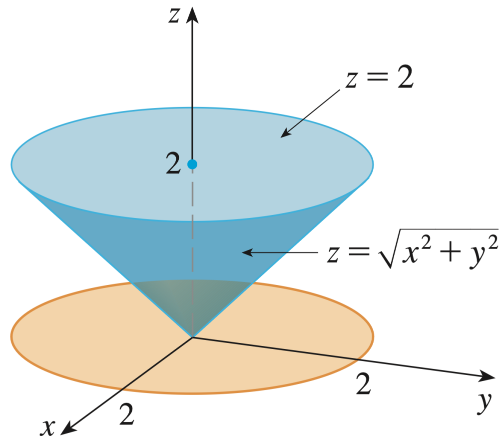




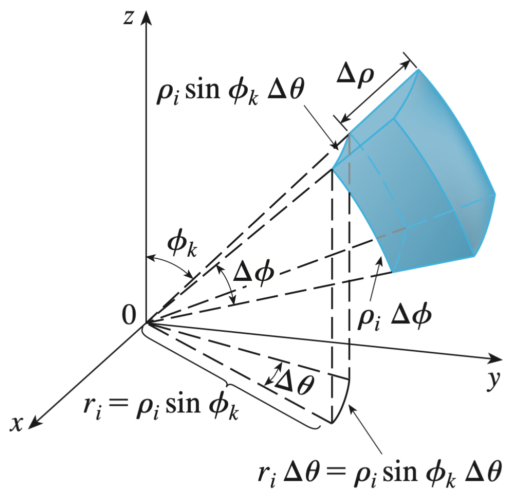
Example. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



Example. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$



Definition. We defined triple integrals by dividing solids into small boxes, but it can be shown that dividing a solid into small spherical wedges always gives the same result. What is the volume of a small spherical wedge E_{ijk}?



Question. How can we think about $\iiint_E f(x,y,z)\,dV$ using spherical coordinates?

Answer.

- If $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$ is a spherical wedge,

$$\iiint\limits_E f(x,y,z)\,dV =$$

- If E is a more general spherical region, such as

$$E = \{(\rho,\theta,\phi) \mid \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)\},$$

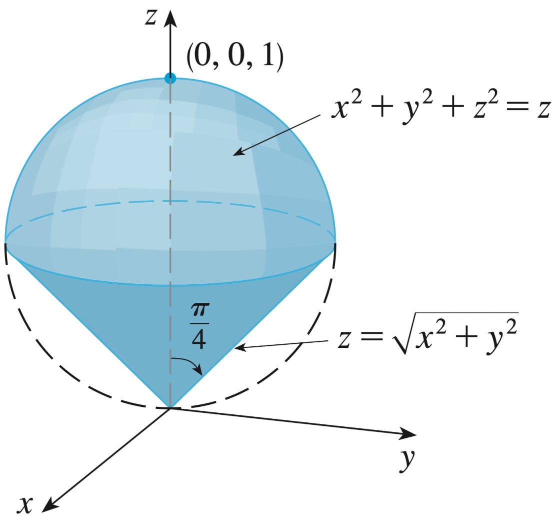
  then

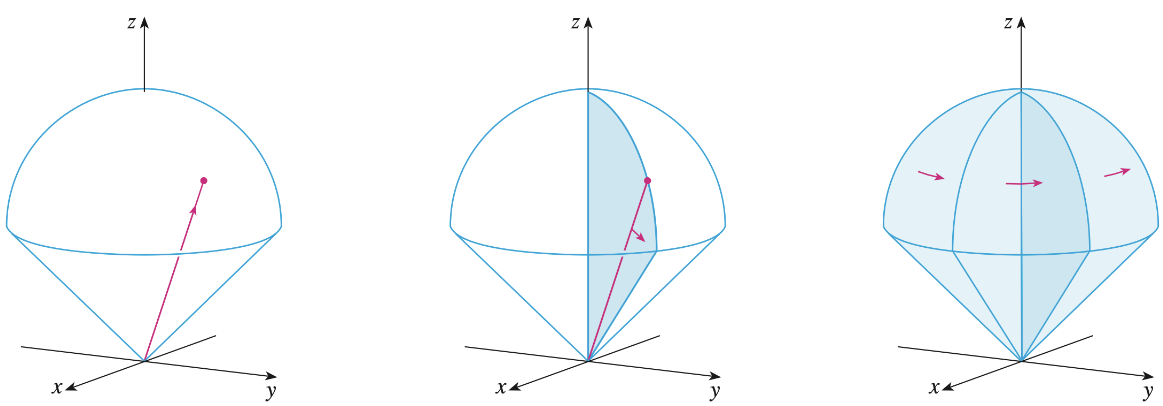
$$\iiint\limits_E f(x,y,z)\,dV =$$

Example. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$$

Example. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.





ρ varies from 0 to $\cos \phi$ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.