Lecture Notes
Math 2400 - Calculus III
Spring 2024

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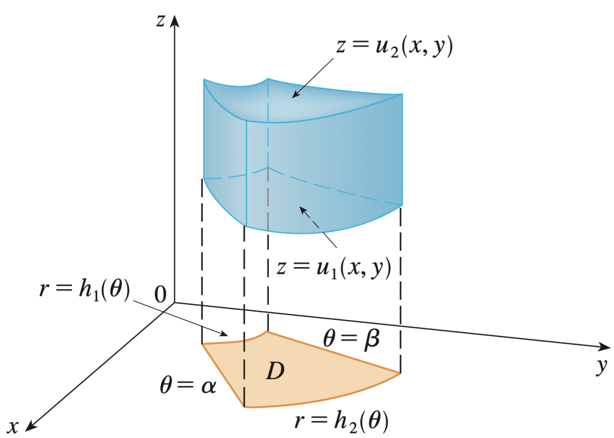
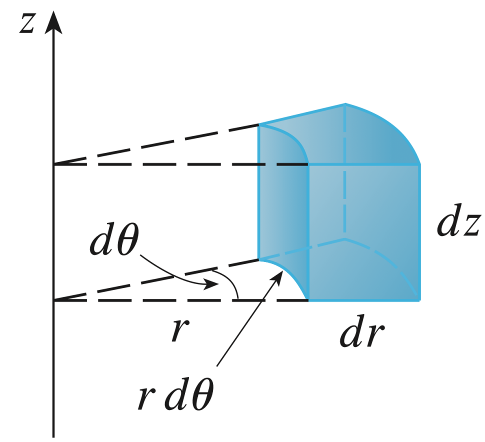
12.8 Triple Integrals in Cylindrical and Spherical Coordinates

Definition. Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates. How can we think about $\iiint_E f(x, y, z)\, dV$ using cylindrical coordinates?





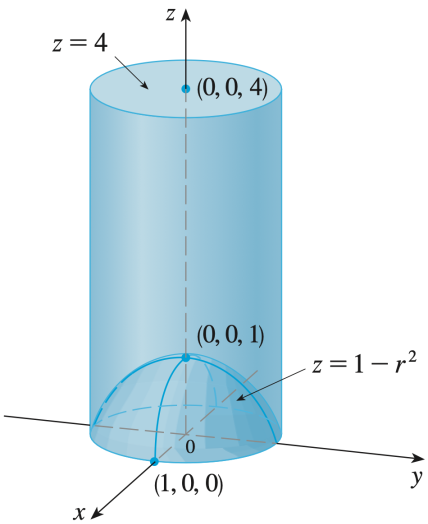
$r\,dr\,d\theta$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) \cdot r\, dz\, dr\, d\theta$$

(Idea: $dV = r\,dz\,dr\,d\theta$)

Example. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



- In cylindrical coordinates, the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$

- So E is the set of (r, θ, z) with

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad 1 - r^2 \le z \le 4$$

- The density function is $f(x,y,z) = K \cdot \sqrt{x^2+y^2} = K \cdot r$ where K is the proportionality constant.

- $$m = \iiint_E K \cdot \sqrt{x^2+y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^{4} (Kr) \cdot r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 Kr^2 \left[4 - (1 - r^2)\right] dr\, d\theta$$

$$= K \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 3r^2 + r^4 \, dr$$

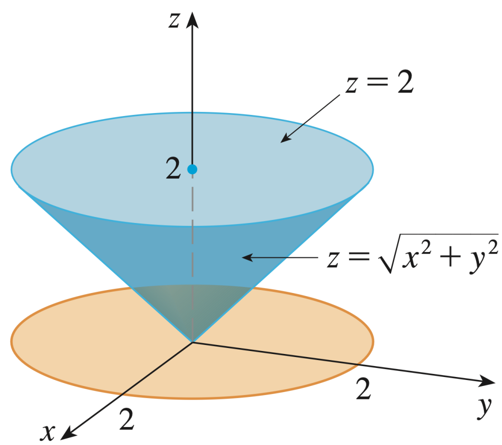
$$= K \cdot 2\pi \cdot \left[r^3 + \frac{r^5}{5}\right]_{r=0}^{r=1}$$

$$= \frac{12\pi K}{5}$$

Example. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$.

- The region E is all points (x,y,z) so that

$$\sqrt{x^2+y^2} \le z \le 2$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$
$$-2 \le x \le 2$$



- The projection of E onto the xy-plane is $x^2+y^2 \le 4$
- In cylindrical coordinates, E is all (r,θ,z) so that

$$r \le z \le 2 \qquad 0 \le r \le 2 \qquad 0 \le \theta \le 2\pi$$

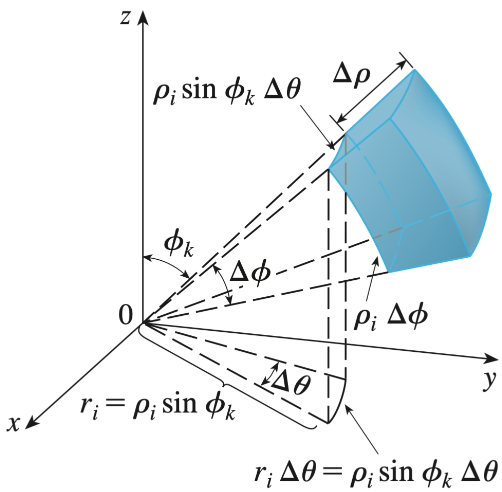
- Hence $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx = \iiint_E (x^2+y^2)\,dV$

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2 \cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 r^3(2-r)\,dr$$

$$= 2\pi \cdot \left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_{r=0}^{r=2} = \frac{16\pi}{5}$$

Definition. We defined triple integrals by dividing solids into small boxes, but it can be shown that dividing a solid into small spherical wedges always gives the same result. What is the volume of a small spherical wedge E_{ijk}?



- Let $E = \overset{\rho}{[a,b]} \times \overset{\theta}{[\alpha,\beta]} \times \overset{\phi}{[c,d]}$ be a spherical wedge
- Divide E into sub-wedges E_{ijk} by subdividing the intervals above.
- If the number of subdivisions is small enough, E_{ijk} is approximately a box, with volume

$$(\Delta\rho)\cdot(\rho_i\Delta\phi)\cdot(\rho_i \sin\phi_k \Delta\theta) = \rho_i^2 \sin\phi_k \overbrace{\Delta\rho\,\Delta\theta\,\Delta\phi}^{\Delta V}$$

Question. How can we think about $\iiint_E f(x,y,z)\,dV$ using spherical coordinates?

Answer.

- If $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$ is a spherical wedge,

$$\iiint\limits_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot\rho^2\sin\phi \, \overbrace{d\rho\,d\theta\,d\phi}^{dV}$$

- If E is a more general spherical region, such as

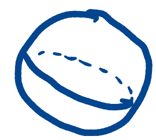
$$E = \{(\rho,\theta,\phi) \mid \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)\},$$

then

$$\iiint\limits_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\cdot\rho^2\sin\phi \, d\rho\,d\theta\,d\phi$$

Example. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$$



Using spherical coordinates,

$$B = \overset{\rho}{[0, 1]} \times \overset{\theta}{[0, 2\pi]} \times \overset{\phi}{[0, \pi]}$$

Therefore,

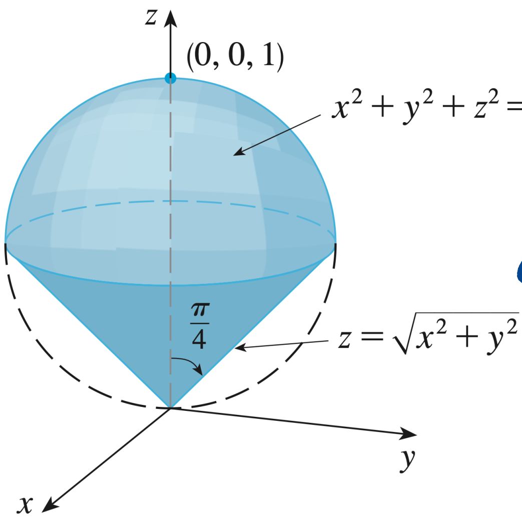
$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \sin\phi \, d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^2 e^{\rho^3} \, d\rho$$

$$= \Big[-\cos\phi \Big]_{\phi=0}^{\phi=\pi} \cdot 2\pi \cdot \Big[\tfrac{1}{3} e^{\rho^3} \Big]_{\rho=0}^{\rho=1}$$

$$= \frac{4}{3}\pi(e - 1)$$

Example. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.



Sphere: $\rho^2 = \rho\cos\phi$

$$\boxed{\rho = \cos\phi}$$

Cone: $\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

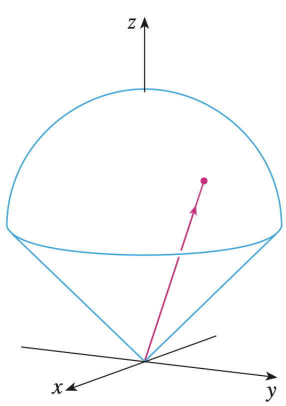
$\rho\cos\phi = \rho\sin\phi$

$\cos\phi = \sin\phi \quad \Rightarrow \quad \boxed{\phi = \frac{\pi}{4}}$

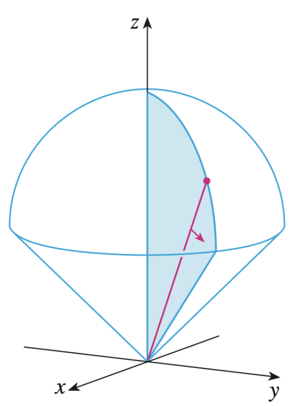
$$V(E) = \iiint_E dV = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi = 2\pi \cdot \int_0^{\pi/4} \sin\phi \cdot \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

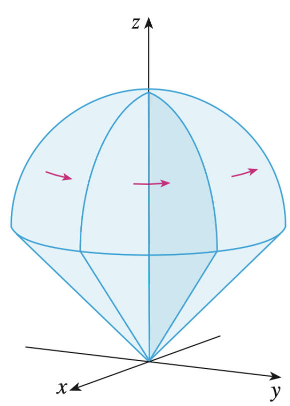
$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_{\phi=0}^{\phi=\pi/4} = \boxed{\frac{\pi}{8}}$$



ρ varies from 0 to $\cos\phi$ while ϕ and θ are constant.



ϕ varies from 0 to $\pi/4$ while θ is constant.



θ varies from 0 to 2π.