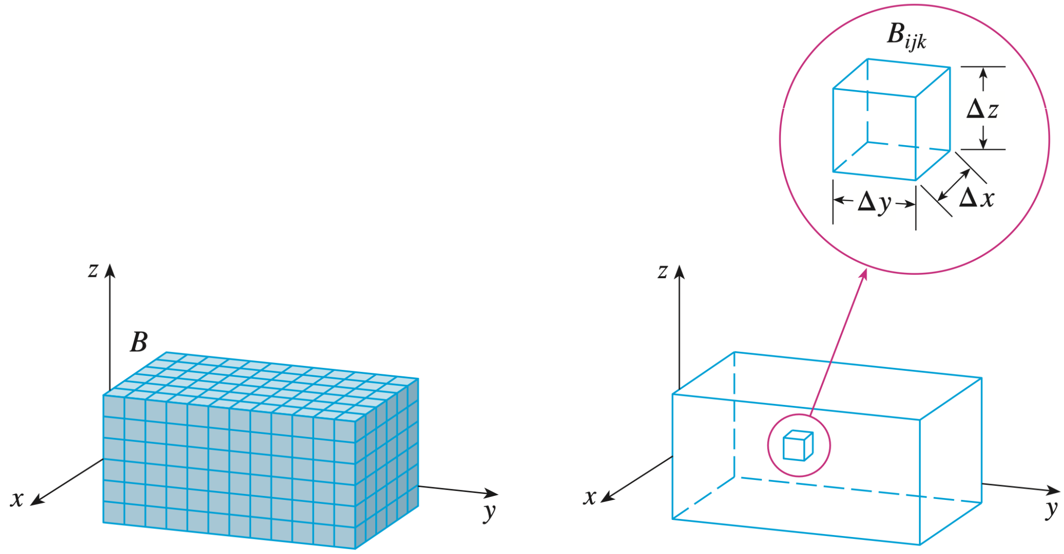
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

12.7 Triple Integrals

Definition. What is the triple integral of f over the box B?

$f(x,y,z)$



- Divide B into sub-boxes B_{ijk}
- Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$
- The triple integral of $f(x,y,z)$ over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

if this limit exists.

Sample point in B_{ijk}

Theorem (Fubini's Theorem for Triple Integrals). If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then how can we evaluate $\iiint_B f(x,y,z)\,dV$ using an iterated integral?

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

(All six possible orders of integration give the same value)

Example. Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is the rectangular box given by

$$B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$$

- We could choose any of the six possible orders of integration
- Let's choose $dx\,dy\,dz$

$$\iiint_B xyz^2\,dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2}{2} \cdot yz^2\right]_{x=0}^{x=1} dy\,dz$$

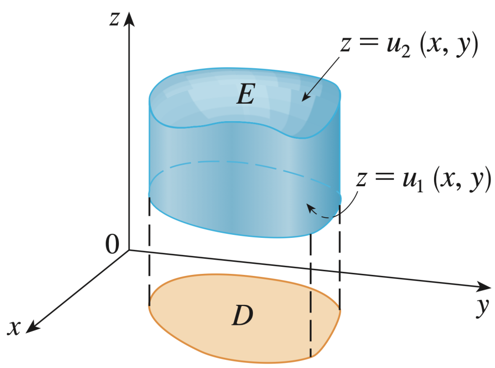
$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\,dy\,dz$$

$$= \int_0^3 \left[\frac{y^2 z^2}{4}\right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\,dz = \left[\frac{z^3}{4}\right]_{z=0}^{z=3} = \frac{27}{4}$$

dz first

Definition. What is a type 1 solid region? If E is type 1, what is $\iiint_E f(x,y,z)\,dV$?



E is a type 1 region if it lies between the graphs of two continuous functions of x and y

- That is, $E = \{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \}$
- If E is type 1, $\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$

  where D is the projection of E to the xy-plane.
- If D is type I, we get

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

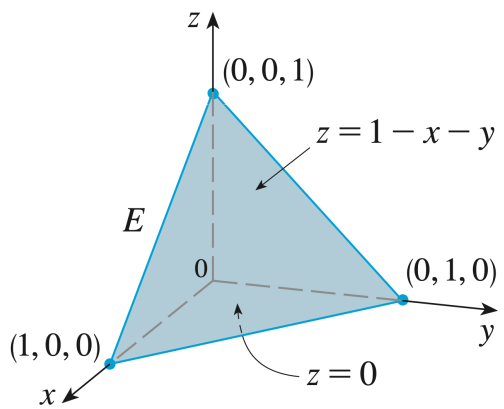
type I

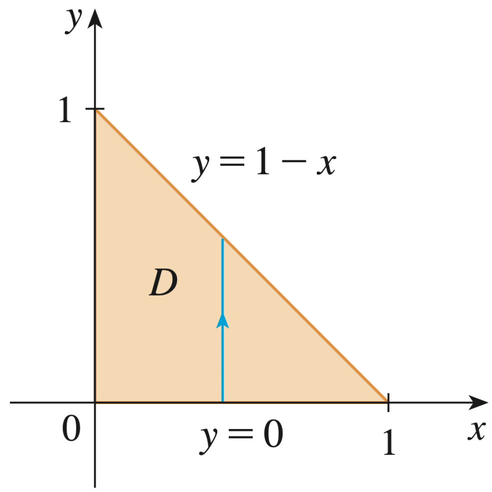
- If D is type II, we get

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

type II

Example. Evaluate $\iiint_E z\,dV$, where E is the ~~solit~~ Sol'd tetrahedron bounded by the four planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.





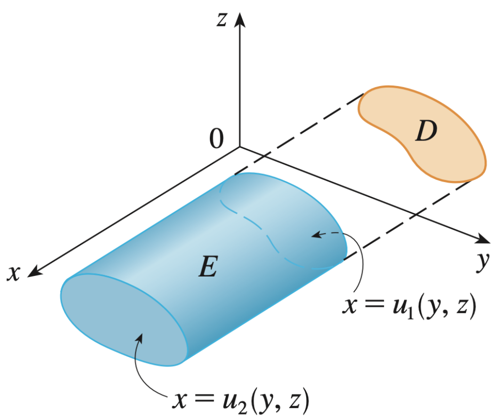
- Lower boundary: $z = 0$
- Upper boundary: $z = 1 - x - y$
- Projection of E to the xy-plane is the triangular region D bounded by $x=0$, $y=0$, and $y = 1-x$
- Viewing D as a type I region, we get

$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$= \frac{1}{24}$$

dx first

Definition. What is a type 2 solid region? If E is type 2, what is $\iiint_E f(x, y, z)\, dV$?



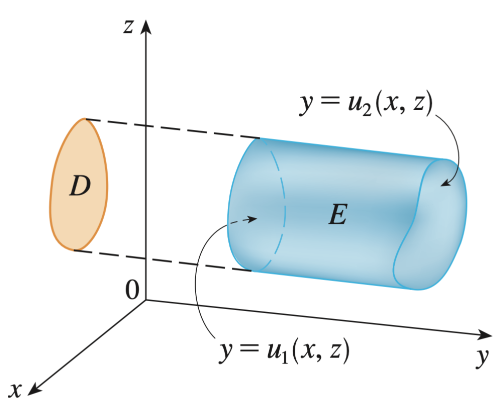
E is type 2 if it lies between functions $u_1(y,z)$ and $u_2(y,z)$

Here, $$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right] dA$$

where D is the projection of E to the yz-plane.

dy first

Definition. What is a type 3 solid region? If E is type 3, what is $\iiint_E f(x, y, z)\, dV$?



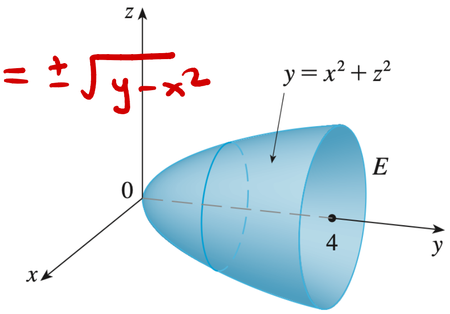
E is type 3 if it lies between functions $u_1(x,z)$ and $u_2(x,z)$

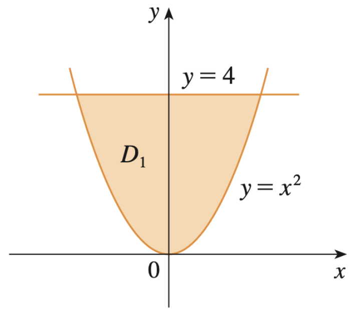
Here, $$\iiint_E f(x,y,z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy \right] dA$$

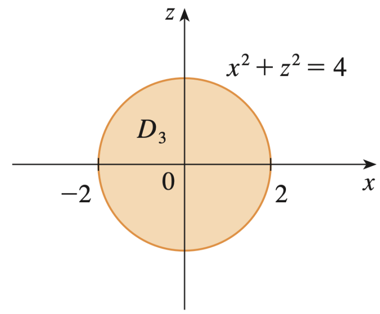
where D is the projection of E to the xz-plane.

Example. Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

$z = \pm\sqrt{y - x^2}$







Type 1?

(dz first)

Lower bound: $z = -\sqrt{y-x^2}$ Upper bound: $z = \sqrt{y-x^2}$

Projection to the xy-plane is D_1

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_1}\left[\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\right] dA$$

difficult $\longrightarrow$ $$= \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx$$

Type 3

(dy first)

Left boundary: $y = x^2 + z^2$ Right boundary: $y = 4$

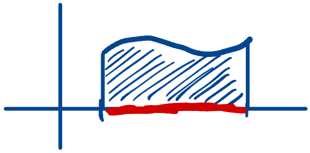
Projection to xz-plane is D_3

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_{D_3}\left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right] dA$$

$$= \iint_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA$$

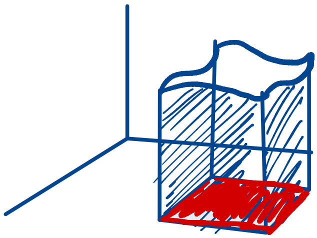
polar in the xz-plane $\longrightarrow$ $$= \int_0^{2\pi}\int_0^2 (4-r^2)\,r\cdot r\,dr\,d\theta = \frac{128\pi}{15}$$

$$\int_a^b f(x)\,dx$$



area under curve

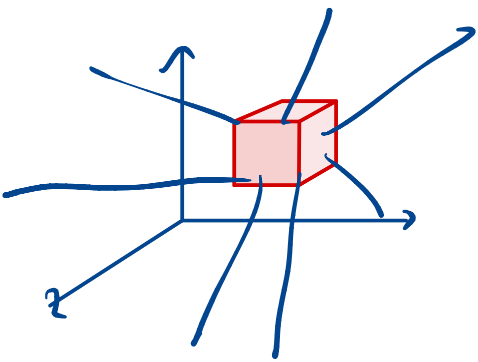
$$\int_a^b\int_c^d f(x,y)\,dy\,dx$$



volume under surface

$$\int_a^b\int_c^d\int_r^s f(x,y,z)\,dz\,dy\,dx$$

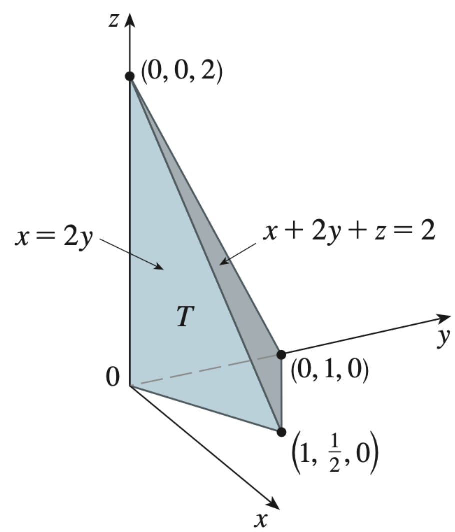
hypervolume under a 4-D graph

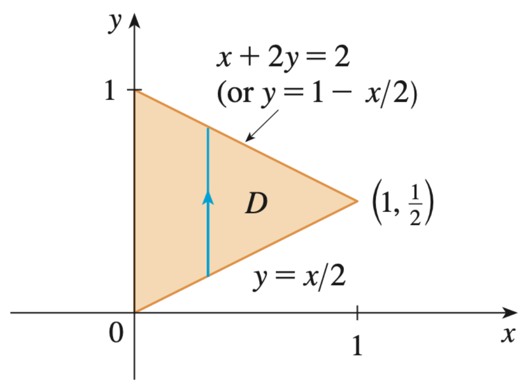


Note: if $f(x,y,z)$ is a density function, the triple integral computes the mass of the region

$$V(E) = \iiint_E 1\,dV$$

Example. Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.





Lower boundary: $z = 0$

Upper boundary: $z = 2 - x - 2y$

The projection D is bounded by $x = 0$, $y = \frac{x}{2}$, and $y = 1 - \frac{x}{2}$

$$V(T) = \iiint_T 1\,dV = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} dz\,dy\,dx$$

$$= \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} 2 - x - 2y\,dy\,dx$$

$$= \frac{1}{3}$$

(See the calculation in §12.3)

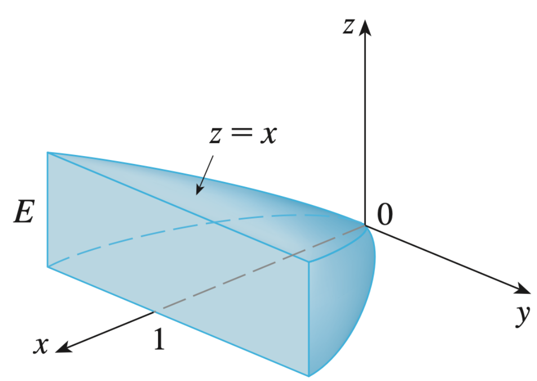
Definition. How can we compute the mass and the center of mass of a solid object that occupies the region E if its density, in units of mass per unit volume, at any given point (x, y, z) is given by $\rho(x, y, z)$?

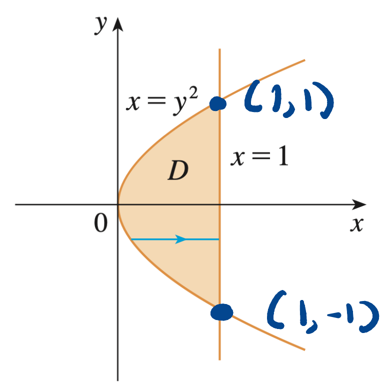
Mass: $m = \iiint_E \rho(x,y,z)\, dV$

Moments: $M_{yz} = \iiint_E x \cdot \rho(x,y,z)\, dV \quad M_{xz} = \iiint_E y\, \rho(x,y,z)\, dV \quad M_{xy} = \iiint_E z \cdot \rho(x,y,z)\, dV$

Center of Mass: $\bar{x} = \dfrac{M_{yz}}{m} \quad \bar{y} = \dfrac{M_{xz}}{m} \quad \bar{z} = \dfrac{M_{xy}}{m}$

Example. Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.





View E as a type 1 region and choose order of integration dz dx dy

① $m = \iiint_E \rho\, dV = \iint_D \left[\int_0^x \rho\, dz \right] dA$

$= \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho\, dz\, dx\, dy$

$= \frac{4}{5} \rho$

② $M_{xy} = \iiint_E z \cdot \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} z \cdot \rho \, dz \, dx \, dy$

$$= \frac{2\rho}{7}$$

$M_{yz} = \iiint_E x \cdot \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} x \cdot \rho \, dz \, dx \, dy$

$$= \frac{4\rho}{7}$$

$M_{xz} = 0$ (E is symmetric about the xz-plane)

③ $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$

$$= \left(\frac{4\rho/7}{4\rho/5}, \frac{0}{4\rho/5}, \frac{2\rho/7}{4\rho/5} \right)$$

$$= \left(\frac{5}{7}, 0, \frac{5}{14} \right)$$