Lecture Notes
Math 2400 - Calculus III
Spring 2024

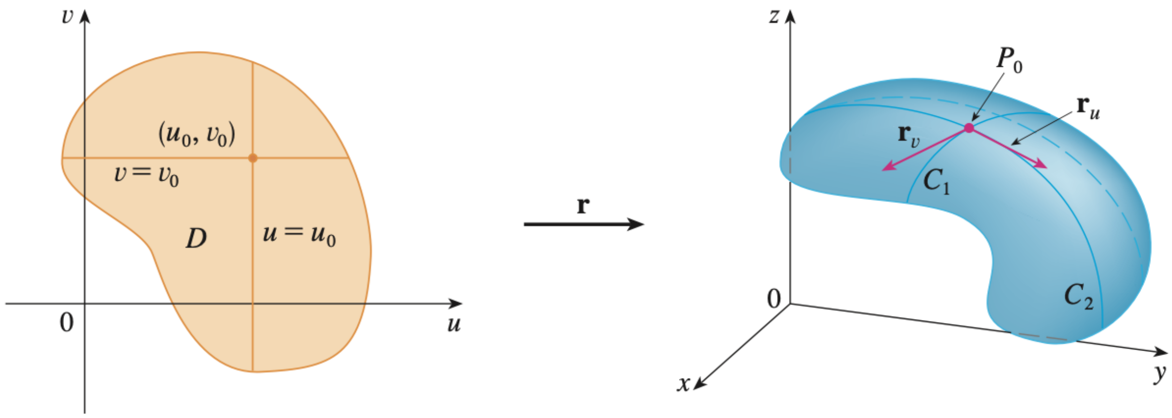
Name: ______________________

12.6 Surface Area

Question. Recall that a parametric surface S is defined by a vector-valued function of two parameters

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

How will we find the area of S?



- We will divide S into patches, and we will approximate each patch by the area of a piece of a tangent plane.

Recall: At $P_0 = \vec{r}(u_0,v_0)$

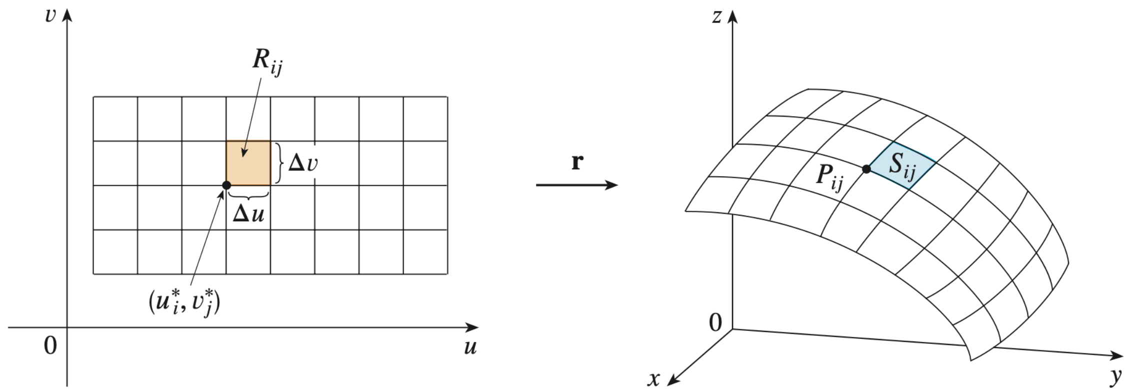
$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right\rangle$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

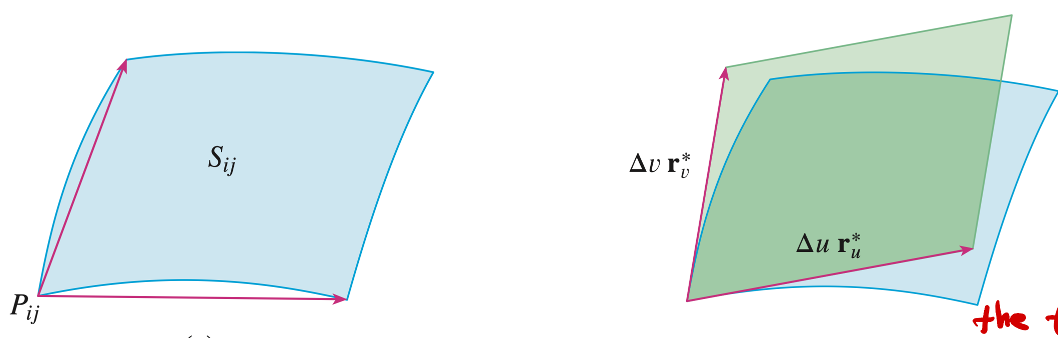
Question. What is a patch?

For simplicity, suppose the parameter domain D is a rectangle.



- Subdivide D into rectangles R_{ij}. Let (u_i^*, v_j^*) be the lower left corner of R_{ij}.
- R_{ij} corresponds to a patch S_{ij} on the surface S.
- If $P_{ij} = \vec{r}(u_i^*, v_j^*)$ is the lower left corner of S_{ij}, the tangent vectors at P_{ij} are $\vec{r}_u^{\,*} = \vec{r}_u(u_i^*, v_j^*)$ and $\vec{r}_v^{\,*} = \vec{r}_v(u_i^*, v_j^*)$.

Question. How can the two edges of the patch that meet at P_{ij} can be approximated by vectors? What is the approximate area of the patch?



(a) (b)

the tangent vectors at P_{ij}

- Each patch S_{ij} is roughly a parallelogram
- The sides of this parallelogram are approximately $\Delta u\, \vec{r}_u^{\,*}$ and $\Delta v\, \vec{r}_v^{\,*}$
- Therefore, the area of the patch is approximately

$$|\Delta u\, \vec{r}_u^{\,*} \times \Delta v\, \vec{r}_v^{\,*}| = |\vec{r}_u^{\,*} \times \vec{r}_v^{\,*}|\, \Delta u\, \Delta v$$

Definition. If a smooth parametric surface S is given by the equation

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

for $(u,v) \in D$, and S is covered just once as (u,v) ranges throughout the parameter domain D, what is the surface area of S?

- Each patch S_{ij} on S has approximate area $|\vec{r}_u^* \times \vec{r}_v^*| \Delta u \Delta v$
- So S has approximate area $\displaystyle\sum_{i=1}^{m} \sum_{j=1}^{n} |\vec{r}_u^* \times \vec{r}_v^*| \Delta u \Delta v$
- As we increase the number of rectangles, we obtain

$$\boxed{A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA}$$

Example. Find the surface area of a sphere of radius a.

① A parametric representation of the sphere is

$$\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$$

where the parameter domain is $D = \overset{\phi}{[0,\pi]} \times \overset{\theta}{[0,2\pi]}$

② $$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} i & j & k \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \langle a^2\sin^2\phi\cos\theta,\ a^2\sin^2\phi\sin\theta,\ a^2\sin\phi\cos\phi \rangle$$

Thus $$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^2\phi} = a^2\sin\phi$$

since $\sin\phi \geq 0$ for $0 \leq \phi \leq \pi$

③ $$A = \iint_D |\vec{r}_\phi \times \vec{r}_\theta|\, dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\, d\phi\, d\theta = a^2\int_0^{2\pi} d\theta \cdot \int_0^{\pi}\sin\phi\, d\phi$$

$$= a^2 \cdot 2\pi \cdot 2 = \boxed{4\pi a^2}$$

Example. How can we find the surface area of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives.

① S has parametric representation

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

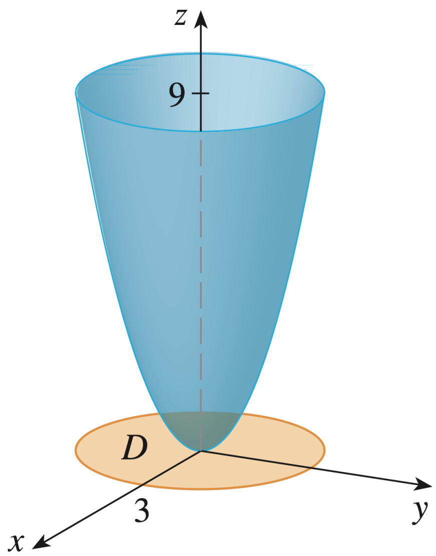
② $\vec{r}_x = \langle 1, 0, f_x \rangle$

$\vec{r}_y = \langle 0, 1, f_y \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

③ $$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

Example. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



The plane $z=9$ intersects $z=x^2+y^2$ in the circle $x^2+y^2=9$, $z=9$. So, the surface lies above the disk D with center the origin and radius 3.

- Hence

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$= \iint_D \sqrt{1 + 4(x^2+y^2)}\, dA$$

- Converting to polar,

$$A = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^3 r\sqrt{1+4r^2}\, dr$$

$$= 2\pi \cdot \left[\frac{1}{8}\cdot\frac{2}{3}(1+4r^2)^{3/2}\right]_{r=0}^{r=3} = \frac{\pi}{6}\left(37\sqrt{37}-1\right)$$