Lecture Notes
Math 2400 - Calculus III
Spring 2024

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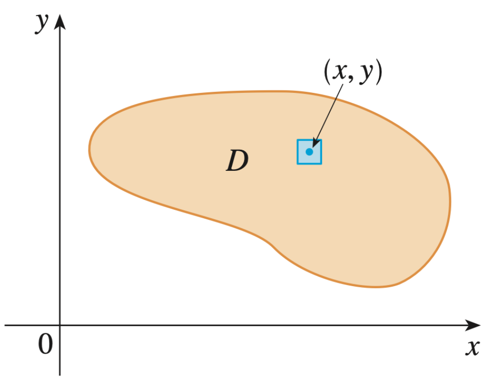
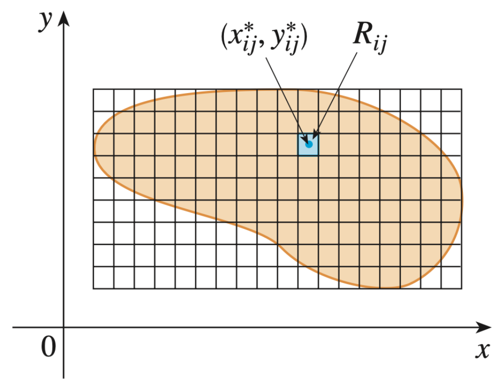
12.5 Applications of Double Integrals

Question. Suppose a lamina (a thin plate) occupies a region D, and its density at a point (x, y) is given by $\rho(x, y)$. How can we compute the total mass of the lamina?

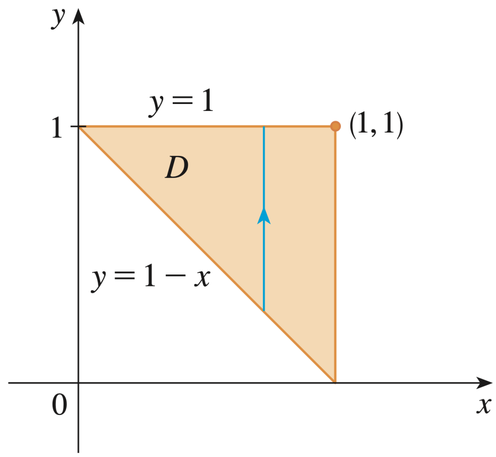




- $\rho(x,y)$ is mass per unit area (e.g. kg/m^2)
- The mass of a subrectangle R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*) \cdot \Delta A$
- The total mass is approximately $m = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \rho(x_{ij}^*, y_{ij}^*) \cdot \Delta A$
- As the # of subrectangles $\to \infty$, $m = \iint_D \rho(x,y)\, dA$

Rmk: Other types of density functions are similar.
If an electric charge is distributed over D with charge density $\sigma(x,y)$, $\iint_D \sigma(x,y)\, dA$ is the total charge.

Example. Charge is distributed over the triangular region D so that the charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.



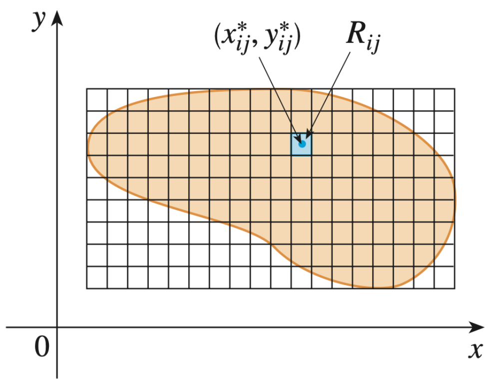
$$Q = \iint_D \sigma(x,y)\, dA = \int_0^1 \int_{1-x}^1 xy \, dy\, dx$$

$$= \int_0^1 \left[x \cdot \frac{y^2}{2} \right]_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2}\left[1^2 - (1-x)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 2x^2 - x^3 \, dx = \frac{1}{2}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{5}{24}$$

The total charge is $\frac{5}{24}$ C

Definition. The moment of a particle about an axis measures the particle's tendency to rotate about that axis. It is computed by taking the product of the particle's mass and its distance from the axis. How can we compute the moment of an entire lamina about the x-axis and the y-axis?



Q: What is the moment of R_{ij} with respect to the x-axis?

A: $$\underbrace{[\rho(x_{ij}^*, y_{ij}^*)\Delta A]}_{\text{mass}} \underbrace{y_{ij}^*}_{\text{distance to the x-axis}}$$

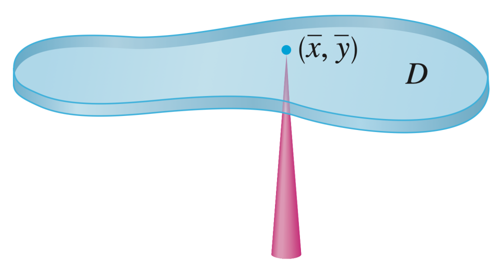
Taking the sum of all of these as the # of rectangles $\to \infty$,

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\,\rho(x,y)\,dA$$

Similarly,

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\,\rho(x,y)\,dA$$

Question. How can we compute the center of mass $(\bar{x}, \bar{y})$ of a lamina that occupies a region D and has density function $\rho(x, y)$?

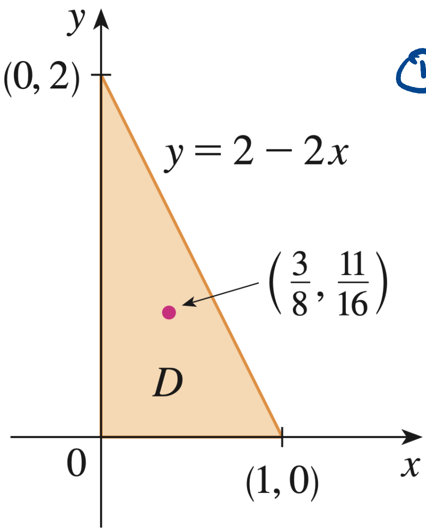


$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

Where the mass m is $\iint_D \rho(x,y)\,dA$

Example. Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$ if the density function is $\rho(x,y) = 1 + 3x + y$.



① $$m = \iint_D \rho(x,y)\,dA = \int_0^1 \int_0^{2-2x} 1+3x+y \; dy\,dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= 4\int_0^1 (1-x^2)\,dx = 4\left[x - \frac{x^3}{3} \right]_{x=0}^{x=1} = \frac{8}{3}$$

② $$\bar{x} = \frac{1}{m} \iint_D x\rho(x,y)\,dA = \frac{3}{8} \int_0^1 \int_0^{2-2x} x + 3x^2 + xy \; dy\,dx$$

$$= \frac{3}{8} \int_0^1 \left[xy + 3x^2y + x\cdot\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{3}{2} \int_0^1 x - x^3 \, dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4} \right]_{x=0}^{x=1} = \frac{3}{8}$$

③ $$\bar{y} = \frac{1}{m} \iint_D y\rho(x,y)\,dA = \frac{3}{8} \int_0^1 \int_0^{2-2x} y + 3xy + y^2 \; dy\,dx$$

$$= \frac{3}{8} \int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2-2x} dx$$

$$= \frac{1}{4} \int_0^1 7 - 9x - 3x^2 + 5x^3 \, dx$$

$$= \frac{1}{4}\left[7x - 9\cdot\frac{x^2}{2} - x^3 + 5\cdot\frac{x^4}{4} \right]_{x=0}^{x=1} = \frac{11}{16}$$

The center of mass is $\left(\frac{3}{8}, \frac{11}{16}\right)$

Example. The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.