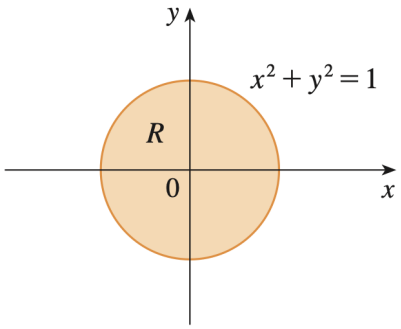
Lecture Notes
Math 2400 - Calculus III
Spring 2024

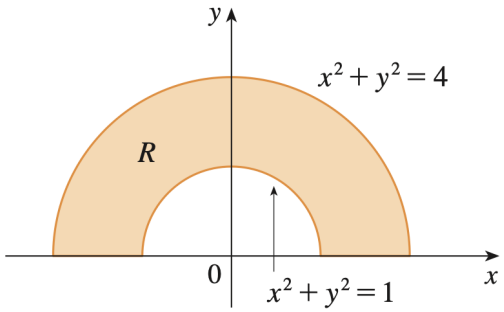
Name: _____

12.4 Double Integrals in Polar Coordinates

Question. What is the goal?



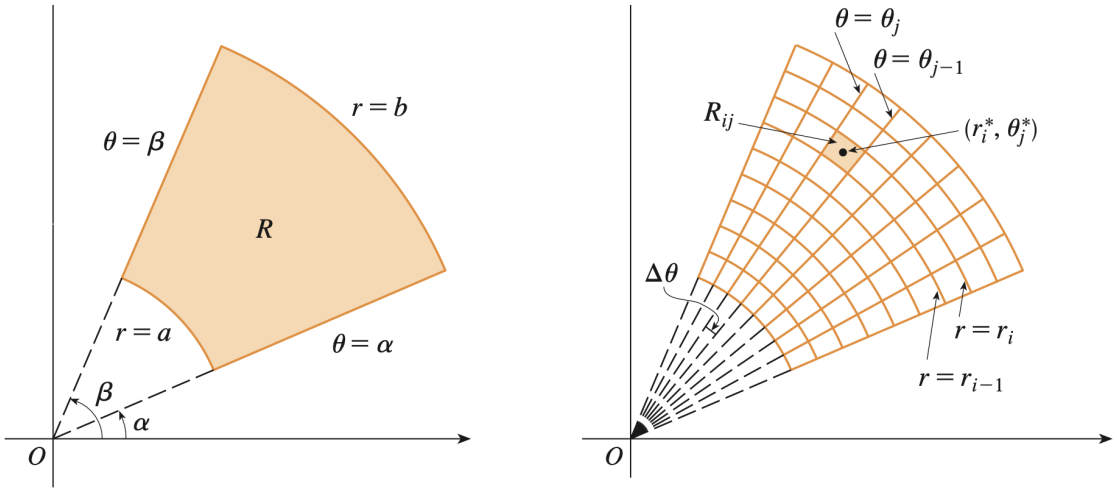
(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Theorem. If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, how can we compute $\iint_R f(x, y)\, dA$?

Question. Consider the following polar rectangle $R = [a, b] \times [\alpha, \beta]$.



To compute $\iint_R f(x, y)\, dA$ using polar coordinates:

- Divide the polar rectange R into polar subrectangles R_{ij}.
- Choose the center (r_i^*, θ_j^*) of R_{ij} as the sample point.
- $\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) \Delta A_i$.

What is the area ΔA_i of R_{ij}?

Question. Using the above, explain why $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

Answer. Let $g(r,\theta) = rf(r\cos\theta, r\sin\theta)$. We have

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i$$

$$=$$

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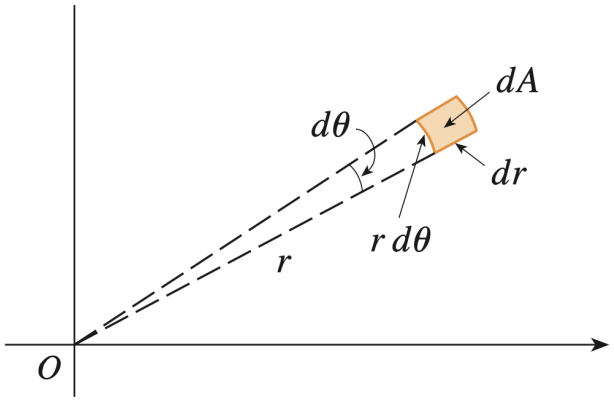
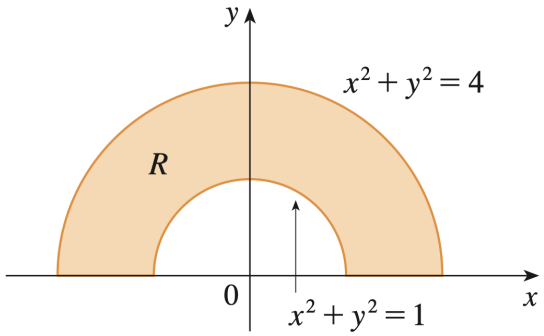
$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

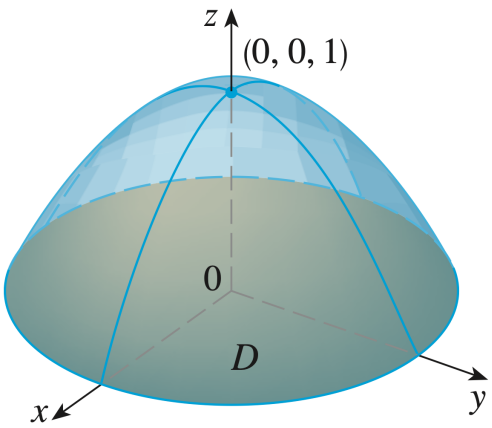
Question. Use the picture below to give an intuitive argument for why dA becomes $rdrd\theta$ when we convert to polar coordinates.



Example. Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



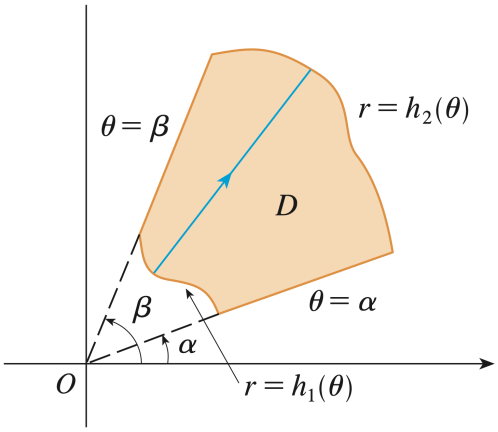
Example. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.



Theorem. If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\},$$

what is $\iint_D f(x, y)\, dA$?



Example. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $(x - 1)^2 + y^2 = 1$.