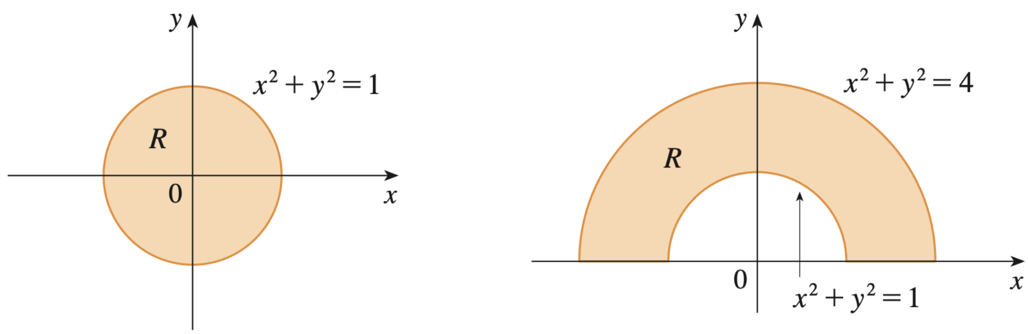
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

12.4 Double Integrals in Polar Coordinates

Question. What is the goal?



(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

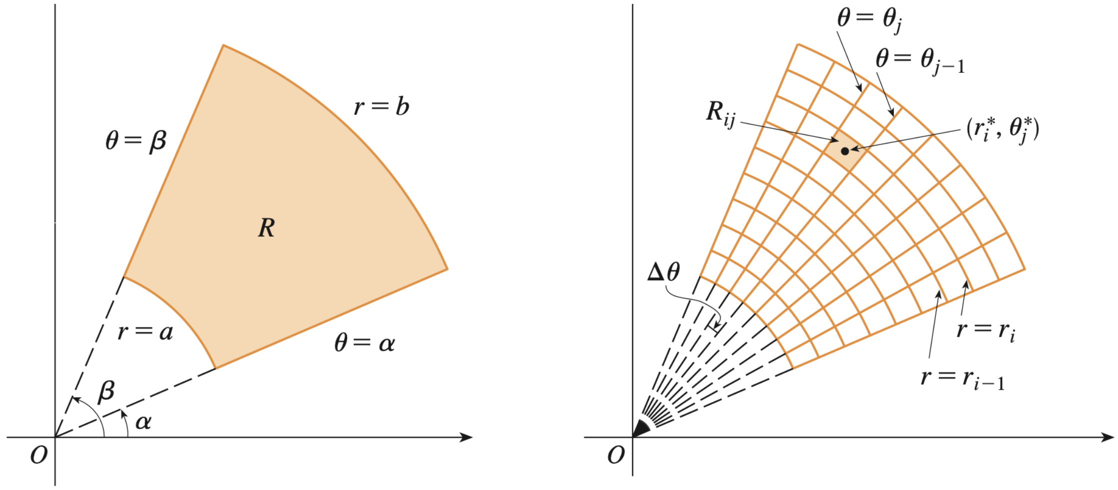
Goal: Evaluate $\iint_R f(x,y)\,dA$ where R is easily described using polar coordinates.

Theorem. If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, how can we compute $\iint_R f(x, y)\, dA$?

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

r θ

Question. Consider the following polar rectangle $R = [a, b] \times [\alpha, \beta]$.



To compute $\iint_R f(x, y)\, dA$ using polar coordinates:

- Divide the polar rectange R into polar subrectangles R_{ij}. → $\Delta r = \frac{b-a}{m}$ $\quad \Delta\theta = \frac{\beta-\alpha}{n}$
- Choose the center (r_i^*, θ_j^*) of R_{ij} as the sample point. → $r_i^* = \frac{1}{2}(r_{i-1} + r_i)$, $\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$
- $\iint_R f(x, y)\, dA = \underbrace{\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n}}_{\text{sum over all columns as the \# } \to \infty} \underbrace{f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)}_{\text{height of the column}} \underbrace{\Delta A_i}_{\text{base of the column}}.$

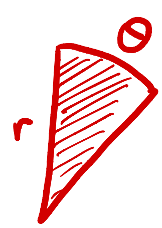
What is the area ΔA_i of R_{ij}?

$$\Delta A_i = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta$$

$$= \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta$$

$$= r_i^* \Delta r \Delta\theta$$



Area of sector is $\frac{1}{2} r^2 \theta$

Question. Using the above, explain why $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

Answer. Let $g(r,\theta) = r f(r\cos\theta, r\sin\theta)$. We have

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta$$

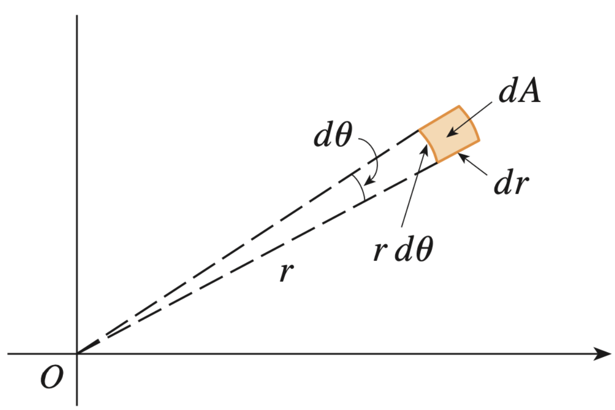
This is a Riemann Sum for using §12.1

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\,\Delta r\,\Delta\theta$$

$$= \int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Question. Use the picture below to give an intuitive argument for why dA becomes $r dr d\theta$ when we convert to polar coordinates.

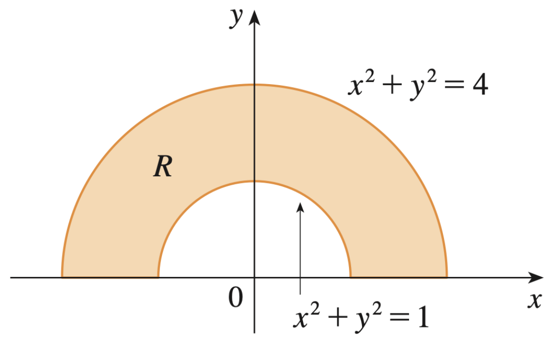


If the polar subrectangles become small enough, they are "essentially" rectangles.

- Base: $r\,d\theta$
- Height: dr

$\Rightarrow dA = r\,dr\,d\theta$

Example. Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



In polar coords, $x^2+y^2=1 \rightsquigarrow r=1$

$x^2+y^2=4 \rightsquigarrow r=2$

$$R = \{(r,\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\} = [1,2] \times [0,\pi]$$

$$\iint_R 3x + 4y^2\, dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi \int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\, dr\, d\theta$$

$$= \int_0^\pi \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2} d\theta$$

$$= \int_0^\pi 7\cos\theta + 15\sin^2\theta\, d\theta$$

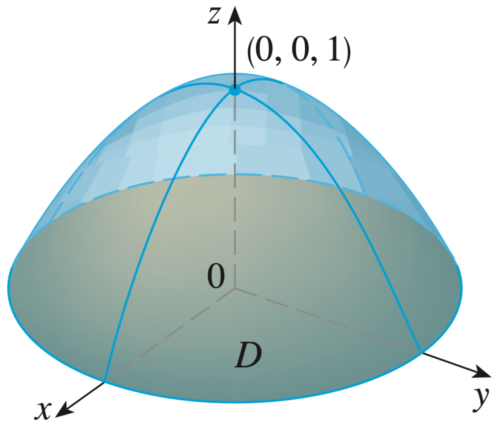
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \int_0^\pi 7\cos\theta + \frac{15}{2}(1-\cos 2\theta)\, d\theta$$

$$= \Big[7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\Big]_{\theta=0}^{\theta=\pi}$$

$$= \frac{15\pi}{2}$$

Example. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.



What is D?

- If $z=0$, $z = 1-x^2-y^2$ becomes $x^2+y^2=1$
- So D is $x^2+y^2 \leq 1$. In polar, this is $[0,1] \times [0, 2\pi]$

So,

$$V = \iint_D (1-x^2-y^2)\, dA = \int_0^{2\pi} \int_0^1 (1-r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r - r^3\, dr = 2\pi \cdot \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=1} = \frac{\pi}{2}$$

If we tried to use rectangular coordinates instead,

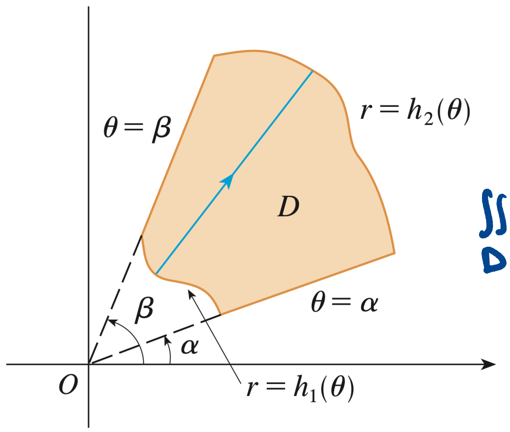
$$V = \iint_D (1-x^2-y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\, dy\, dx$$

This is not so easy.

Theorem. If f is continuous on a polar region of the form

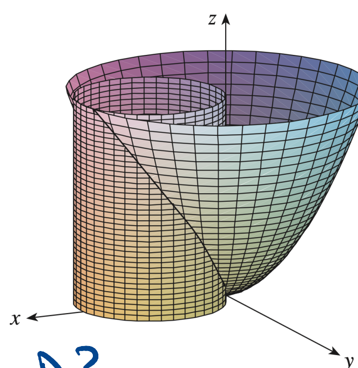
$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\},$$

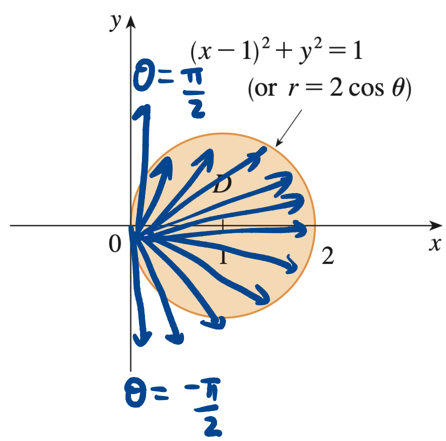
what is $\iint_D f(x, y)\, dA$?



$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $(x-1)^2 + y^2 = 1$.





What is D?

$(x-1)^2 + y^2 = 1$

$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$

$r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = 1$

$r^2 - 2r\cos\theta = 0$

$r - 2\cos\theta = 0$

$r = 2\cos\theta$

$$\boxed{D = \left\{(r,\theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta\right\}}$$

$$V = \iint_D (x^2+y^2)\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r^2)\ r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{4} r^4 \right]_{r=0}^{r=2\cos\theta} d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4} \cdot 16 \cos^4\theta\ d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= 4 \int_{-\pi/2}^{\pi/2} \cos^4\theta\ d\theta$$

$$= 4 \int_{-\pi/2}^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\ d$$

$$= \int_{-\pi/2}^{\pi/2} 1 + 2\cos 2\theta + \left(\frac{1+\cos 4\theta}{2} \right) d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\ d\theta$$

$$= \left[\frac{3}{2}\theta \right]_{-\pi/2}^{\pi/2} + \cancelto{0}{\left[\sin 2\theta \right]_{-\pi/2}^{\pi/2}} + \cancelto{0}{\left[\frac{1}{8}\sin 4\theta \right]_{-\pi/2}^{\pi/2}}$$

$$= \boxed{\frac{3\pi}{2}}$$