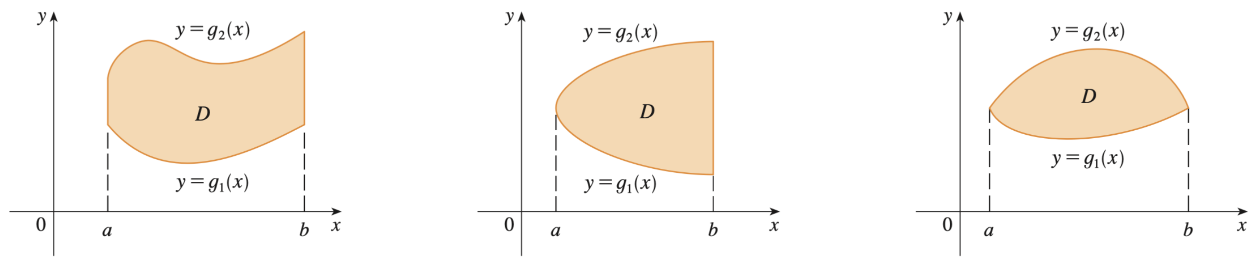
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

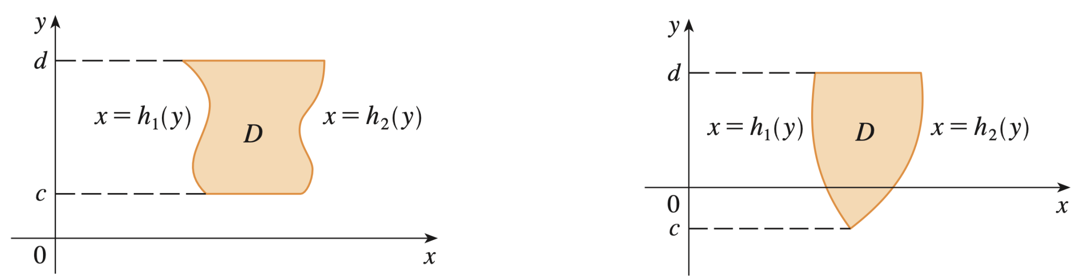
## 12.3 Double Integrals over General Regions

**Definition.** What is a type I region? If $f$ is continuous on a type I region $D$, how can we evaluate the double integral of $f$ over this region?



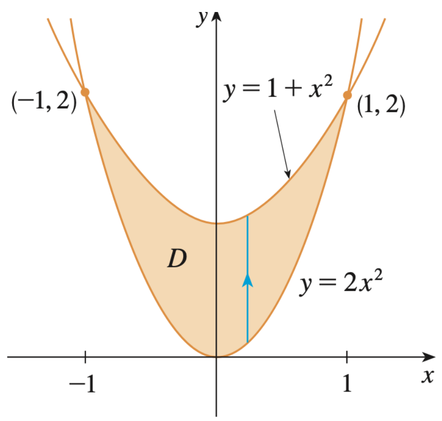

- $D$ is type I if it lies between the graphs of two continuous functions of $x$, i.e. $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$
- In this case, $\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

**Definition.** What is a type II region? If $f$ is continuous on a type II region $D$, how can we evaluate the double integral of $f$ over this region?




- $D$ is type II if it lies between the graphs of two continuous functions of $y$, i.e. $D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$
- In this case, $\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

**Example.** Evaluate $\iint_D (x + 2y)\, dA$, where $D$ is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.




① Where do the curves intersect?

$$2x^2 = 1 + x^2$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

$$\Rightarrow (-1, 2) \text{ and } (1, 2)$$

② This is a type I region, not a type II region.
(it helps to draw an arrow between the curves)

↑ = type I

→ = type II

③
$$\iint_D x + 2y \, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y \, dy \, dx$$

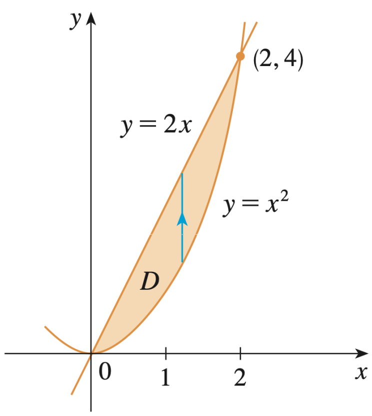
$$= \int_{-1}^{1} \Big[ xy + y^2 \Big]_{y=2x^2}^{1+x^2} dx$$

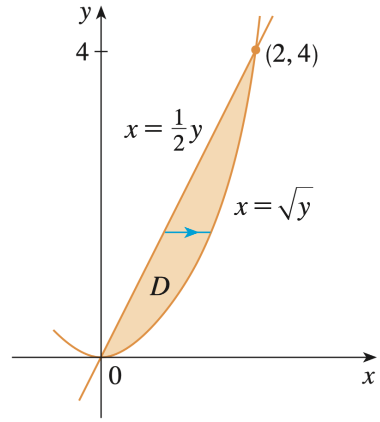
$$= \int_{-1}^{1} x(1+x^2) + (1+x^2)^2 - x(2x^2) - (2x^2)^2 \, dx$$

$$= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1 \, dx$$

$$= \left[ -3 \cdot \frac{x^5}{5} - \frac{x^4}{4} + 2 \cdot \frac{x^3}{3} + \frac{x^2}{2} + x \right]_{x=-1}^{x=1} = \frac{32}{15}$$

**Example.** Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region $D$ in the $xy$-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.




- This region is both type I and type II

Type I

$$V = \int_0^2 \int_{x^2}^{2x} x^2 + y^2 \, dy \, dx = \int_0^2 \left[ x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 x^2(2x) + \frac{(2x)^3}{3} - x^2(x^2) - \frac{(x^2)^3}{3} \, dx$$

$$= \int_0^2 -\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \, dx = \left[ \frac{-x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \right]_{x=0}^{x=2} = \frac{216}{35}$$

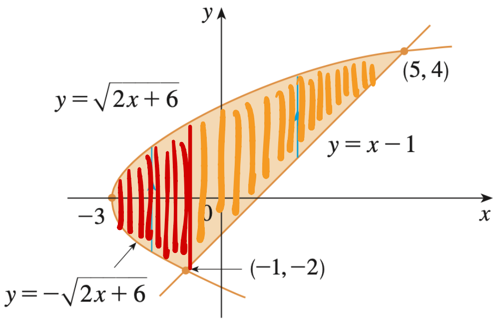
Type II

$$V = \int_0^4 \int_{1/2y}^{\sqrt{y}} x^2 + y^2 \, dx \, dy = \int_0^4 \left[ \frac{x^3}{3} + y^2 x \right]_{x=1/2y}^{x=\sqrt{y}} dy$$

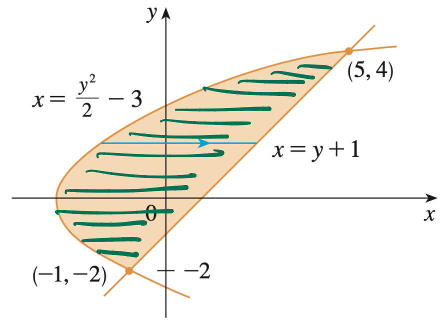
$$= \int_0^4 \frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2} \, dy$$

$$= \left[ \frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \right]_{y=0}^{y=4} = \frac{216}{35}$$

**Example.** Evaluate $\iint_D xy\,dA$, where $D$ is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.




(a) $D$ as a type I region




(b) $D$ as a type II region

Type I

$$\iint_D xy\,dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

Type II

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy$$

$$= \int_{-2}^{4} \left[\frac{x^2}{2}\cdot y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

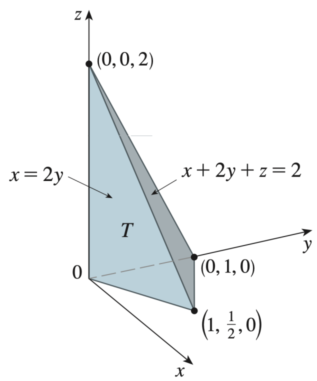
Expand

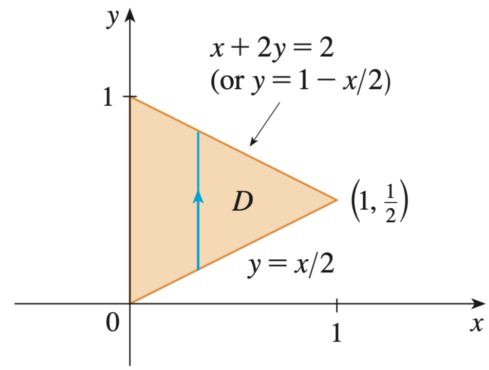
$$= \int_{-2}^{4} \frac{1}{2}(y+1)^2\cdot y - \frac{1}{2}\cdot\left(\frac{1}{2}y^2-3\right)^2\cdot y\,dy$$

$$= \frac{1}{2}\int_{-2}^{4} -\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\,dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\cdot\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36$$

**Example.** Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.




- What is $D$? The planes $x=0$, $x=2y$, and $x+2y+z=2$ intersect $z=0$ in the lines $x=0$, $x=2y$, and $x+2y=2$

$$V = \iint_D 2-x-2y \, dA$$

$$= \int_0^1 \int_{x/2}^{1-x/2} 2-x-2y \, dy \, dx$$

$$= \int_0^1 \Big[ 2y - xy - y^2 \Big]_{y=\frac{x}{2}}^{y=1-\frac{x}{2}} dx$$

Simplify

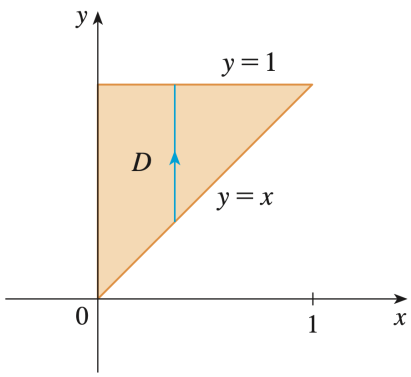
$$= \int_0^1 2\left(1-\frac{x}{2}\right) - x\left(1-\frac{x}{2}\right) - \left(1-\frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4} \, dx$$

$$= \int_0^1 x^2 - 2x + 1 \, dx$$

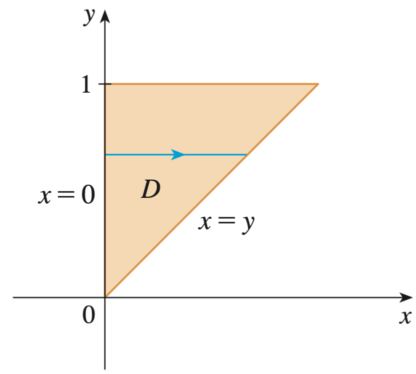
$$= \left[ \frac{x^3}{3} - x^2 + x \right]_0^1 = \frac{1}{3}$$

★ Good example to study

**Example.** Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy dx$.




(Type I)




(Type II)

- Hard to compute $\int \sin(y^2)\, dy$ (actually impossible)
- View D as a type II region instead.
- $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$

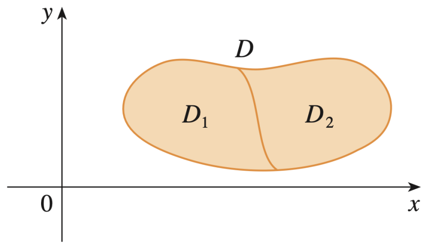
$$= \int_0^1 \left[ x \sin(y^2) \right]_{x=0}^{x=y} dy$$

$$= \int_0^1 y \sin(y^2)\, dy$$

$$= \left[ -\frac{1}{2} \cos(y^2) \right]_{y=0}^{y=1}$$

$$= \frac{1}{2}\left(1 - \cos(1)\right)$$

**Question.** Suppose $D = D_1 \cup D_2$, where $D_1$ and $D_2$ don't overlap except perhaps on their boundaries. How can we evaluate $\iint_D f(x,y)\, dA$?




$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

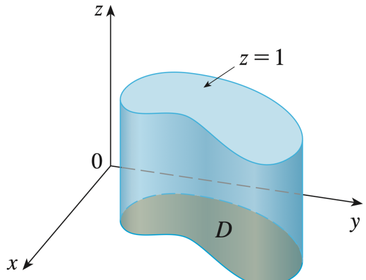
**Question.** How can we use the above to evaluate double integrals over regions $D$ that are neither type I nor type II?


(a) $D$ is neither type I nor type II.

(b) $D = D_1 \cup D_2$; $D_1$ is type I, $D_2$ is type II.

- Write $D$ as a union of regions of type I or type II
- Use the above

**Question.** How can we use a double integral to compute the area of a region $D$?




Area of $D = \iint_D 1\, dA$

↖ "cylinder" with base $D$ and height $1$