Lecture Notes
Math 2400 - Calculus III
Spring 2024

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12.2 Iterated Integrals

Question. How can we evaluate a double integral of a function $f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ by using iterated integrals?

- We will see that

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

- To evaluate $\int_c^d f(x,y)\,dy$, we integrate $f(x,y)$ from $y=c$ to $y=d$ while regarding x as a constant. (this is called "partial integration with respect to y")

- This produces a function $A(x) = \int_c^d f(x,y)\,dy$ that is a function of x only. We then evaluate $\int_a^b A(x)\,dx$.

Example. Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

(a) Regarding x as a constant,

$$\int_1^2 x^2 y \, dy = \left[x^2 \cdot \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \cdot \left(\frac{2^2}{2} \right) - x^2 \cdot \left(\frac{1^2}{2} \right) = \frac{3}{2} x^2$$

We now integrate this function from $x=0$ to $x=3$

$$\int_0^3 \frac{3}{2} x^2 \, dx = \left[\frac{x^3}{2} \right]_{x=0}^{x=3} = \frac{27}{2}$$

(b) In this case, we first regard y as a constant

$$\int_0^3 x^2 y \, dx = \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} = 9y$$

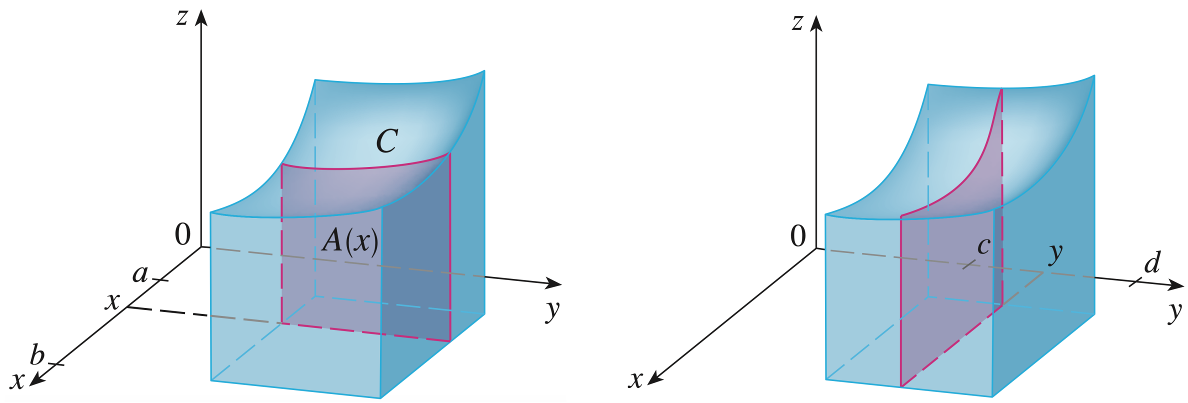
We now integrate this function from $y=1$ to $y=2$

$$\int_1^2 9y \, dy = \left[9 \cdot \frac{y^2}{2} \right]_{y=1}^{y=2} = \frac{36}{2} - \frac{9}{2} = \frac{27}{2}$$

Theorem (Fubini's Theorem). If f is continuous on the rectangle $R = [a, b] \times [c, d]$, what does Fubini's Theorem say about $\iint_R f(x, y)\, dA$?

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Proof.



- Proved in graduate real analysis using measure theory
- Idea: If we fix some x, we can compute

$$A(x) = \int_c^d f(x,y)\, dy$$

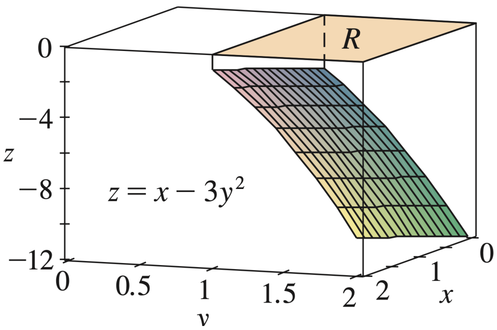
$$= \text{area of the cross-section at } x$$

Now integrate these cross-sections $A(x)$ from $x=a$ to $x=b$ to get the total volume.

- Fubini says this is the same as looking at cross-sections parallel to the x-axis and integrating those from $y=c$ to $y=d$.

□

Example. Use Fubini's theorem to evaluate $\iint_R (x - 3y^2)\, dA$, where $R = [0,2] \times [1,2]$.



① $$\iint_R x - 3y^2 \, dA = \int_0^2 \left[\int_1^2 x - 3y^2 \, dy \right] dx = \int_0^2 \left[xy - y^3 \right]_{y=1}^{y=2} dx$$

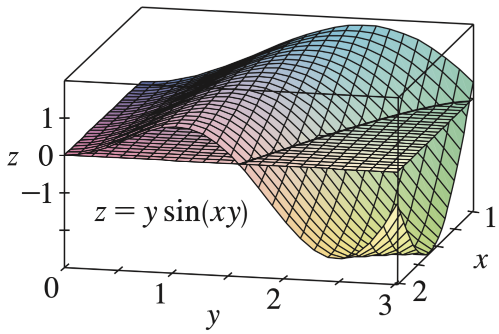
$$= \int_0^2 x - 7 \, dx = \left[\frac{x^2}{2} - 7x \right]_{x=0}^{x=2} = -12$$

② $$\iint_R x - 3y^2 \, dA = \int_1^2 \left[\int_0^2 x - 3y^2 \, dx \right] dy = \int_1^2 \left[\frac{x^2}{2} - 3y^2 x \right]_{x=0}^{x=2} dy$$

$$= \int_1^2 2 - 6y^2 \, dy = \left[2y - 2y^3 \right]_{y=1}^{y=2} = -12$$

(We estimated this to be −11.875 in §12.1)

Example. Use Fubini's theorem to evaluate $\iint_R y\sin(xy)\,dA$ where $R = [1,2]\times[0,\pi]$.



① If we first integrate with respect to x,

$$\int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy$$

$$= \int_0^\pi \Big[-\cos(xy)\Big]_{x=1}^{x=2} dy = \int_0^\pi -\cos(2y) + \cos(y)\,dy$$

$$= \Big[-\tfrac{1}{2}\sin(2y) + \sin(y)\Big]_{y=0}^{y=\pi} = 0$$

② If we first integrate with respect to y, we get

$$\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$$

We can evaluate $\int_0^\pi y\sin(xy)\,dy$ using integration by parts.

$$u = y \qquad dv = \sin(xy)\,dy$$

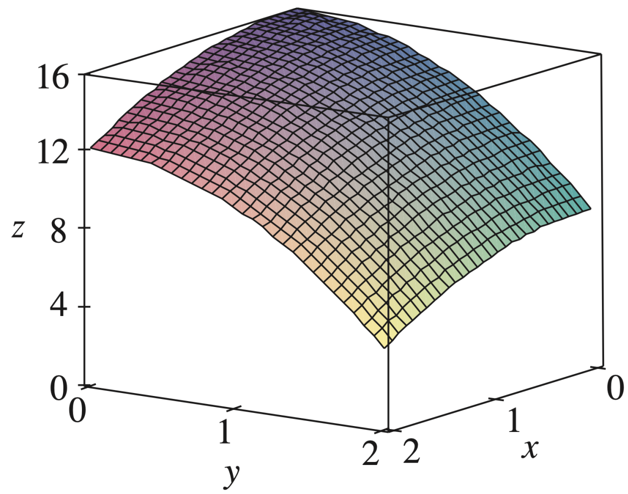
$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

Eventually you get $-\dfrac{\pi\cos\pi x}{x} + \dfrac{\sin\pi x}{x^2}$

To integrate this from $x=1$ to $x=2$, need integration by parts...

Point: Sometimes one order of integration is much easier than the other

Example. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.



S lies under $z = 16 - x^2 - 2y^2$ and above the rectangle $R = [0,2] \times [0,2]$

$$V = \iint_R 16 - x^2 - 2y^2 \, dA = \int_0^2 \int_0^2 16 - x^2 - 2y^2 \, dx \, dy$$

$$= \int_0^2 \Big[16x - \tfrac{1}{3}x^3 - 2y^2 x \Big]_{x=0}^{x=2} dy$$

$$= \int_0^2 \frac{88}{3} - 4y^2 \, dy$$

$$= \Big[\frac{88}{3} y - \frac{4}{3} y^3 \Big]_{y=0}^{y=2} = 48$$

(Compare this to our estimate in §12.1)

Theorem. If $f(x,y)$ can be factored as the product of a function of x only and a function of y only, how can we rewrite the double integral of f over a rectangle $R = [a,b] \times [c,d]$?

If $f(x,y) = g(x) \cdot h(y)$, then

$$\iint_R f(x,y)\, dA = \iint_R g(x) \cdot h(y)\, dA = \int_a^b g(x)\, dx \cdot \int_c^d h(y)\, dy$$

Proof.

$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_a^b g(x)\, h(y)\, dx \right] dy$$

h(y) is a constant in the inner integral

$$= \int_c^d h(y) \left[\int_a^b g(x)\, dx \right] dy$$

$\int_a^b g(x)\, dx$ is a constant

$$= \int_a^b g(x)\, dx \cdot \int_c^d h(y)\, dy$$

□

Example. Evaluate the integral $\iint_R \sin x \cos y\, dA$ if $R = [0, \pi/2] \times [0, \pi/2]$.

$$\iint_R \sin x \cos y\, dA = \int_0^{\pi/2} \sin x\, dx \cdot \int_0^{\pi/2} \cos y\, dy$$

$$= \Big[-\cos x \Big]_{x=0}^{x=\pi/2} \cdot \Big[\sin y \Big]_{y=0}^{y=\pi/2}$$

$$= 1 \cdot 1 = 1$$