Lecture Notes
Math 2400 - Calculus III
Spring 2024

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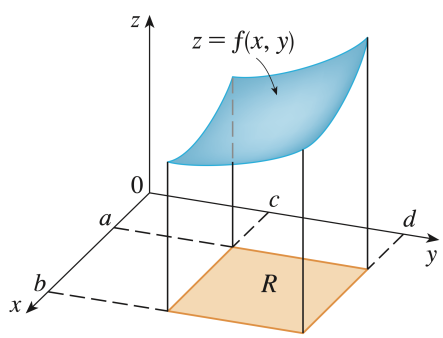
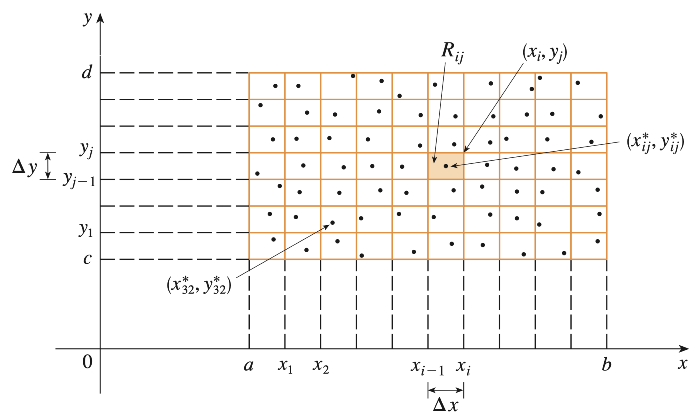
12.1 Double Integrals over Rectangles

Question. Consider a function f defined on a closed rectangle $R = [a, b] \times [c, d]$, and suppose that $f(x, y) \geq 0$. What is the volume of the solid S that lies above R and under the graph of f?





- Divide R into subrectangles:

  Divide $[a,b]$ into m subintervals of width $\Delta x = \frac{b-a}{m}$

  Divide $[c,d]$ into n subintervals of width $\Delta y = \frac{d-c}{n}$

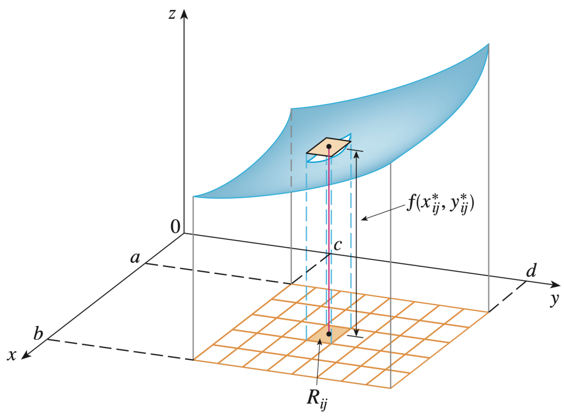
- The subrectangle R_{ij} is $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ and has area $\Delta A = \Delta x \Delta y$

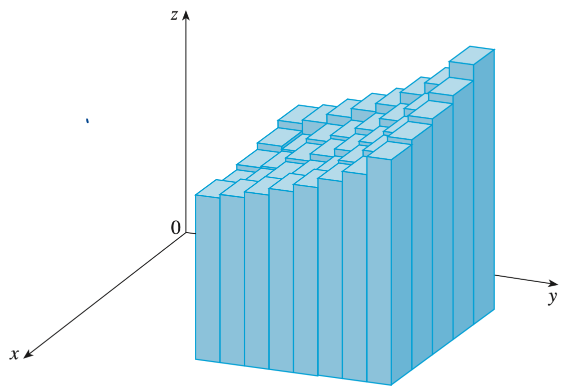
- Choose a sample point (x^*_{ij}, y^*_{ij}) in each R_{ij}

- The volume of the "column" above R_{ij} is approximately $f(x^*_{ij}, y^*_{ij})\,\Delta A$

- Therefore, the total volume under f is approximately

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta A$$



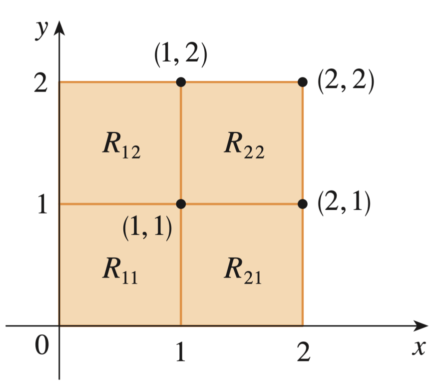


Definition. What is the double integral of f over the rectangle R?

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

Example. Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}.

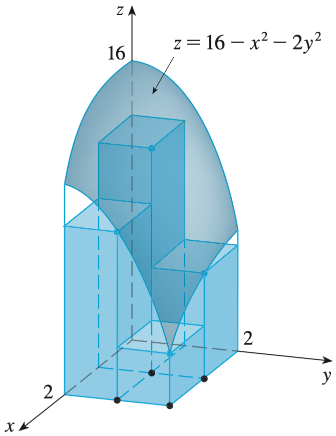


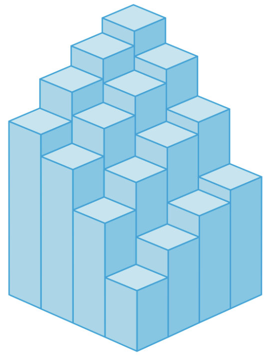
$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Here, $m = n = 2$ and $\Delta A = 1$

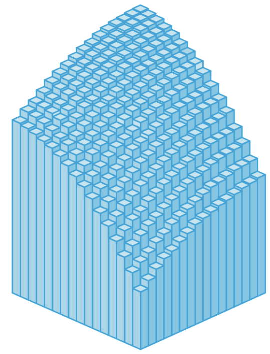
$\Rightarrow f(1,1)\cdot 1 + f(1,2)\cdot 1 + f(2,1)\cdot 1 + f(2,2)\cdot 1$

$\Rightarrow 13\cdot 1 + 7\cdot 1 + 10\cdot 1 + 4\cdot 1 = \boxed{34}$





$m = n = 4, V \approx 41.5$

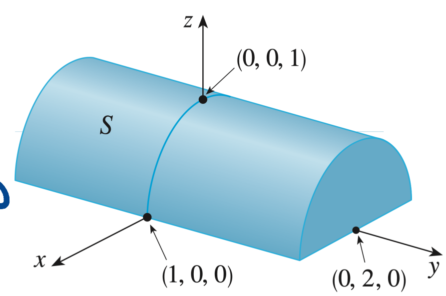


$m = n = 16, V \approx 46.46875$

Example. If $R = \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

by interpreting it geometrically.



- Hard to compute from limit definition

- Note: $z = \sqrt{1 - x^2} \Rightarrow z^2 = 1 - x^2, \ (z \geq 0)$

$\Rightarrow x^2 + z^2 = 1$

- The double integral represents the volume below (half of) a circular cylinder of radius 1

$$V = \frac{1}{2} \cdot (\pi r^2 h) = \frac{1}{2} \cdot (\pi \cdot 1^2 \cdot 4) = 2\pi$$

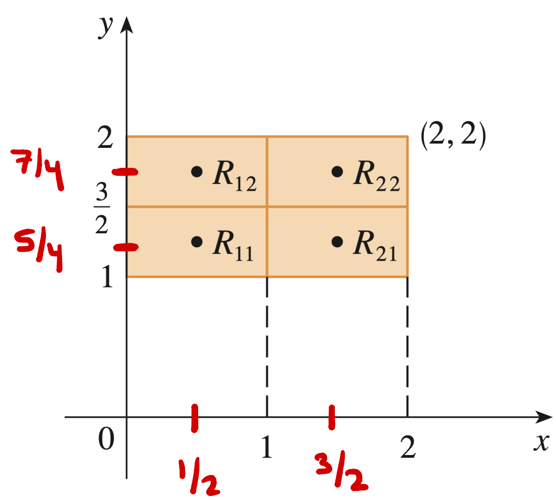
Definition. What is the Midpoint Rule for double integrals?

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and

$\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$

Example. Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)\, dA$, where $R = [0,2] \times [1,2]$.



$$\iint_R (x-3y^2) \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

$\bar{x}_1 = \frac{1}{2}$ $\bar{x}_2 = \frac{3}{2}$

$\bar{y}_1 = \frac{5}{4}$ $\bar{y}_2 = \frac{7}{4}$

$\Delta A = \frac{1}{2}$

$$= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) \cdot \tfrac{1}{2} + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) \cdot \tfrac{1}{2} + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) \cdot \tfrac{1}{2} + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \cdot \tfrac{1}{2}$$

$$= \left(\frac{-67}{16}\right) \cdot \frac{1}{2} + \left(\frac{-139}{16}\right) \cdot \frac{1}{2} + \left(\frac{-51}{16}\right) \cdot \frac{1}{2} + \left(\frac{-123}{16}\right) \cdot \frac{1}{2}$$

$$= \frac{-95}{8} = -11.875$$

Definition. What is the average value of a function $f(x, y)$ on a rectangle R?

$$f_{avg} = \frac{1}{|R|} \iint_R f(x,y)\, dA$$

where $|R|$ means the area of R.

(Intuition: if $z = f(x,y)$ describes a mountainous region, we can chop off the tops at height f_{avg} and fill in the valleys to make the region flat.)

Remark. Properties of Double Integrals

- $\iint\limits_R [f(x,y) + g(x,y)]\, dA =$ $\iint\limits_R f(x,y)\, dA + \iint\limits_R g(x,y)\, dA$
- $\iint\limits_R cf(x,y)\, dA =$ $c \iint\limits_R f(x,y)\, dA$
- What can we say if $f(x,y) \geq g(x,y)$ for all (x,y) in R? $\iint\limits_R f(x,y)\, dA \geq \iint\limits_R g(x,y)\, dA$