Lecture Notes
Math 2400 - Calculus III
Spring 2024

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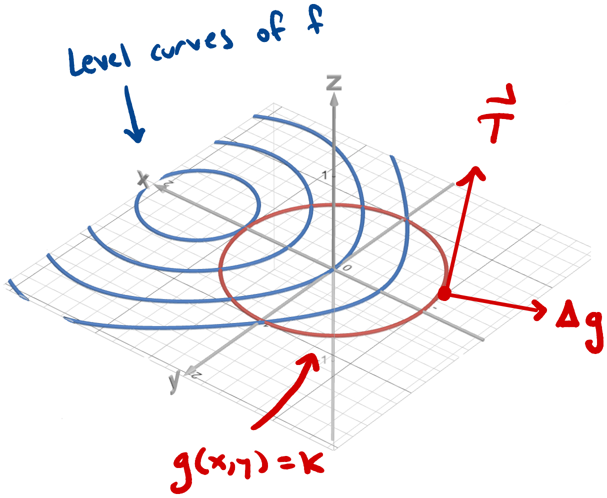
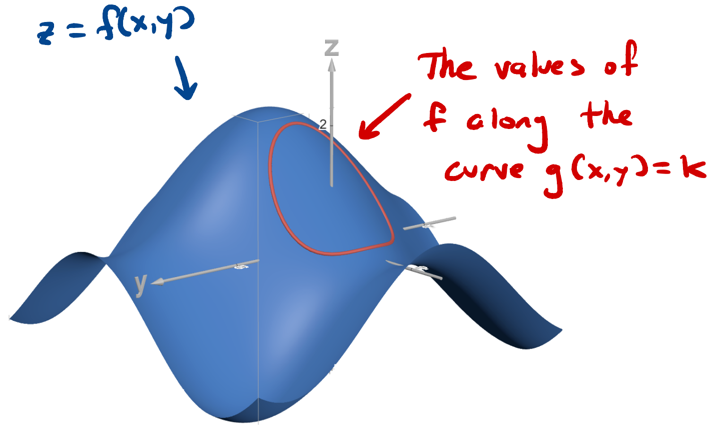
11.8 Lagrange Multipliers

Question. What is the idea of Lagrange Multipliers?





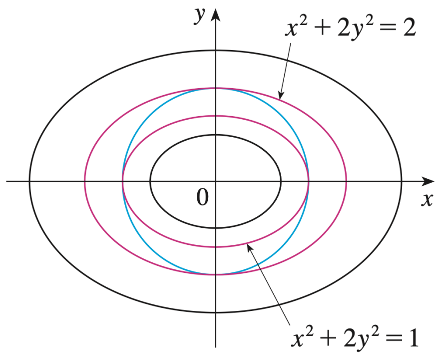
- Goal: Find the maximum (or minimum) of a function $f(x,y)$ along a curve $g(x,y)=k$.

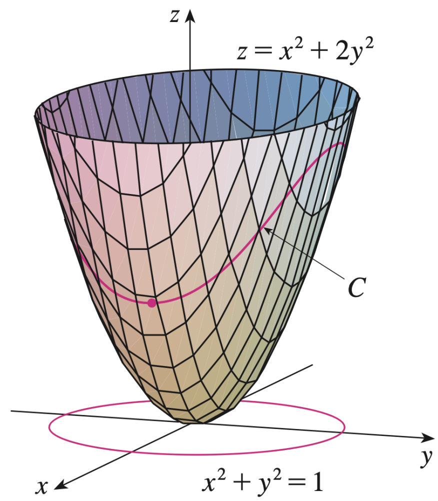
- As we move along the curve, we can compute the directional derivative in the direction of the curve (i.e. along the unit tangent vector $\vec{u} = \frac{\vec{T}}{|\vec{T}|}$)

- A (local) maximum of f along the curve would occur when this directional derivative is 0. That is, $\nabla f \cdot \vec{u} = 0$

- If $\nabla f \cdot \vec{u} = 0$, then ∇f is perpendicular to the curve g. This means the gradient vectors are parallel. $\boxed{\text{So } \nabla f = \lambda \nabla g}$

Example. Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.





- Let $g(x,y) = x^2 + y^2$. Constraint: $g(x,y) = 1$
- We need to solve for $\nabla f = \lambda \nabla g$.
- Since $\nabla f = \langle 2x, 4y \rangle$ and $\nabla g = \langle 2x, 2y \rangle$, we get a system

$$\underset{\text{EQ1}}{2x = 2x \cdot \lambda} \qquad \underset{\text{EQ2}}{4y = 2y \cdot \lambda} \qquad \underset{\text{EQ3}}{x^2 + y^2 = 1}$$

- From EQ1, $x = 0$ or $\lambda = 1$
- If $x = 0$, then EQ3 implies $y = \pm 1$
- If $\lambda = 1$, then EQ2 implies $y = 0$. From EQ3, $x = \pm 1$.
- Hence extreme values are possible at $(0,1), (0,-1), (1,0), (-1,0)$
- $f(0,1) = 2$, $f(0,-1) = 2$ $f(1,0) = 1$ $f(-1,0) = 1$
- Maximum Value: $f(0, \pm 1) = 2$ Minimum Value: $f(\pm 1, 0) = 1$

Example. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

- The disk $x^2+y^2 \leq 1$ is a closed, bounded set, so the Extreme Value Theorem applies.

① Critical points

$f_x = 2x$ and $f_y = 4y$ $\Rightarrow$ one critical point at $(0,0)$

$f(0,0) = 0$

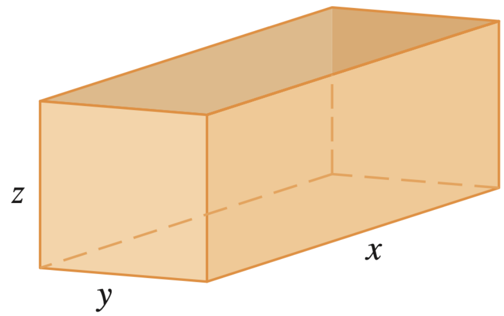
② Extreme values on the boundary: (see previous example)

$f(\pm 1, 0) = 1$ $\quad\quad$ $f(0, \pm 1) = 2$

Conclude: Absolute maximum is $f(0, \pm 1) = 2$

Absolute minimum is $f(0,0) = 0$

Example. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.



- We want to maximize $V(x,y,z) = xyz$ subject to the constraint $g(x,y,z) = 2xz + 2yz + xy = 12$

- Since $\nabla V = \langle yz, xz, xy \rangle$ and $\nabla g = \langle 2z+y, 2z+x, 2x+2y \rangle$ we get a system

EQ1 $yz = \lambda(2z+y)$

EQ2 $xz = \lambda(2z+x)$

EQ3 $xy = \lambda(2x+2y)$

EQ4 $2xz + 2yz + xy = 12$

- Multiply EQ1 by x, EQ2 by y, EQ3 by z

EQ1 $xyz = \lambda(2xz + xy)$

EQ2 $xyz = \lambda(2yz + xy)$

EQ3 $xyz = \lambda(2xz + 2yz)$

EQ4 $2xz + 2yz + xy = 12$

- Observation: $\lambda \neq 0$. ($\lambda = 0$ would mean $xy = xz = yz = 0$, contradicting EQ4)

- From EQ1 and EQ2

since $z \neq 0$

$$2xz + xy = 2yz + xy \Rightarrow xz = yz \Rightarrow x = y$$

- From EQ2 and EQ3

since $x \neq 0$

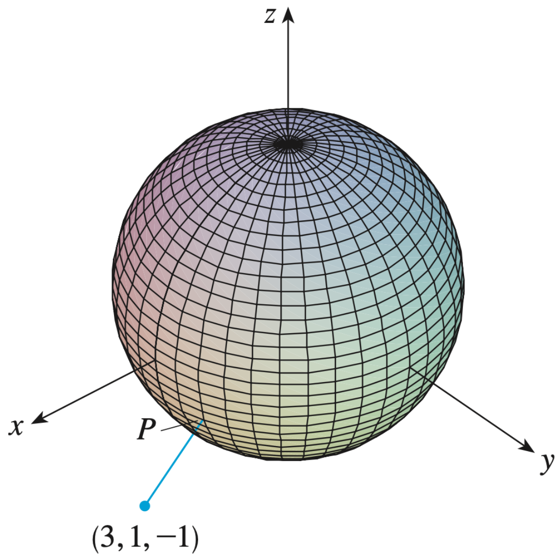
$$2yz + xy = 2xz + 2yz \Rightarrow xy = 2xz \Rightarrow y = 2z$$

- Substituting $x = y = 2z$ into EQ4

$$4z^2 + 4z^2 + 4z^2 = 12$$

$\Rightarrow z = 1$, so $x = 2$ and $y = 2$.

Example. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.



- We will minimize $d^2 = f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$ subject to the constraint $g(x,y,z) = x^2 + y^2 + z^2 = 4$
- Since $\nabla f = \langle 2(x-3), 2(y-1), 2(z+1) \rangle$ and $\nabla g = \langle 2x, 2y, 2z \rangle$ we get a system

EQ1 $\quad 2(x-3) = 2x \cdot \lambda$

EQ2 $\quad 2(y-1) = 2y \cdot \lambda$

EQ3 $\quad 2(z+1) = 2z \cdot \lambda$

EQ4 $\quad x^2 + y^2 + z^2 = 4$

- Solve for x, y, z in terms of λ

$$x = \frac{3}{1-\lambda} \qquad y = \frac{1}{1-\lambda} \qquad z = \frac{-1}{1-\lambda}$$

Note: $\lambda \neq 1$ from, say, EQ1.

· Substituting into EQ4,

$$\left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 = 4$$

and we solve to get $(1-\lambda)^2 = \frac{11}{4} \Rightarrow \lambda = 1 \pm \frac{\sqrt{11}}{2}$

· These values of λ correspond to the points

$$\underbrace{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)}_{\text{closest}} \quad \text{and} \quad \underbrace{\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}_{\text{farthest}}$$

(... check the value of f at these points)

Method of Lagrange Multipliers

Theorem (Two Variables)**.** To find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = k$ (assuming that these extreme values exist and $\Delta g \neq 0$ on the curve $g(x, y) = k$):

(a) Find all values of x, y, and λ such that

$$\Delta f(x, y) = \lambda \Delta g(x, y)$$

and

$$g(x, y) = k$$

(b) Evaluate f at all the points x, y that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Writing the vector equation $\Delta f = \lambda \Delta g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

which is a system of three equations in the three unknowns x, y, and λ. It is not necessary to find explicit values for λ.

Theorem (Three Variables)**.** To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\Delta g \neq 0$ on the surface $g(x, y, z) = k$):

(a) Find all values of x, y, z, and λ such that

$$\Delta f(x, y, z) = \lambda \Delta g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points x, y, z that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Writing the vector equation $\Delta f = \lambda \Delta g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

which is a system of four equations in the four unknowns x, y, z, and λ. It is not necessary to find explicit values for λ.