Lecture Notes
Math 2400 - Calculus III
Spring 2024

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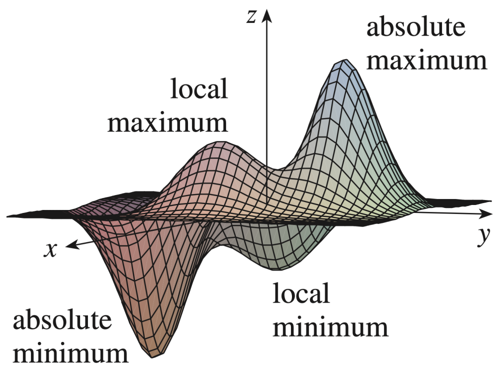
11.7 Maximum and Minimum Values

Question. What is a local maximum? Local minimum? Absolute maximum? Absolute minimum?



Local max at (a,b): $f(x,y) \leq f(a,b)$ for all (x,y) near (a,b)

Local min at (a,b): $f(x,y) \geq f(a,b)$ for all (x,y) near (a,b)

Absolute max: $f(x,y) \leq f(a,b)$ for all (x,y) in the domain of f

Absolute min: $f(x,y) \geq f(a,b)$ for all (x,y) in the domain of f

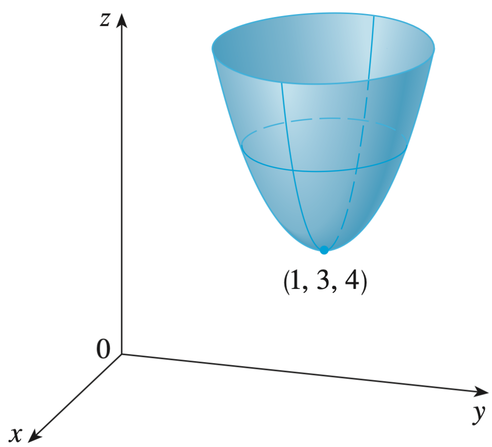
Theorem (Fermat's Theorem)**.** If f has a local maximum or minimum at (a,b), what can we say about the partial derivatives there?

If $f_x(a,b)$ and $f_y(a,b)$ both exist, they are both 0.

Definition. What is a critical point?

- The point (a,b) is a critical point if $f_x(a,b)=0$ and $f_y(a,b)=0$ or if one of these partial derivatives does not exist.
- Idea: Any local max or min occurs at a critical point.
- ⚠ Not all critical points give rise to maxima/minima. Could be neither.

Example. Find the extreme values of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$.



① Find the critical points:

$f_x = 2x - 2 \qquad f_y = 2y - 6$

- These are never DNE
- These are 0 when $x = 1$, $y = 3$

$\Rightarrow$ $(1,3)$ is the only critical point

② Classify $(1,3)$.

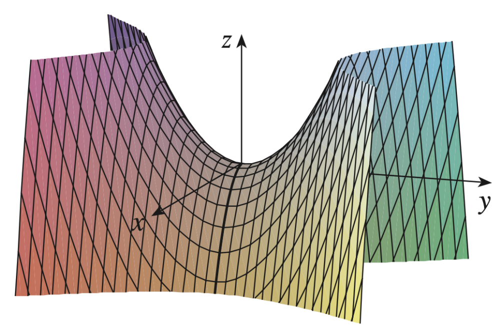
- Completing the square, $f(x,y) = 4 + (x-1)^2 + (y-3)^2$
- Since $(x-1)^2 \geq 0$ and $(y-3)^2 \geq 0$, we have $f(x,y) \geq 4$
- Hence $f(1,3) = 4$ is a local (and absolute) minimum

Example. Find the extreme values of $f(x,y) = y^2 - x^2$.

① Find the critical points:

$f_x = -2x \qquad f_y = 2y$

$\Rightarrow$ The only critical point is $(0,0)$



② Classify $(0,0)$:

- $f(0,0) = 0$
- For points on the x-axis: $f(x,y) = f(x,0) = -x^2 < 0$ (if $x \neq 0$)
- For points on the y-axis: $f(x,y) = f(0,y) = y^2 > 0$ (if $y \neq 0$)

Conclude: every disk with center $(0,0)$ contains points where f is positive and negative. So $f(0,0)$ is neither a max nor a min. (it is a saddle point).

Theorem (Second Derivatives Test)**.** Define the Hessian matrix of $f(x, y)$ to be

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

How can we use the Hessian matrix to determine if f has a local maximum or local minimum at a point (a, b)?

Clairaut's Thm

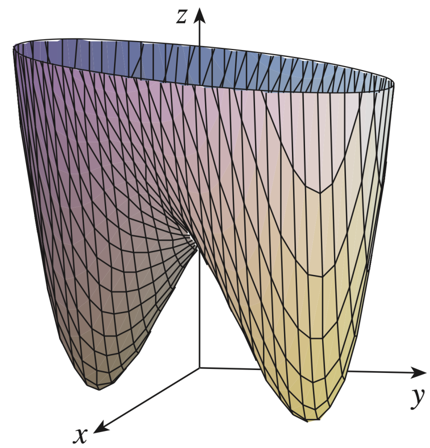
- Define $D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$

$$\begin{tikzcd}
 & & \text{local minimum} \\
 & f_{xx}(a,b) \arrow[ur, ">0"] \arrow[r, "<0"'] & \text{local maximum} \\
D(a,b) \arrow[ur, ">0"] \arrow[r, "<0"] \arrow[dr, "=0"'] & \text{saddle point} & \\
 & \text{no information} &
\end{tikzcd}$$

Example. Find the local maximum and minimum values and saddle points of the function

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

① Find the critical points:



$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

Hence we need to solve:

$$x^3 - y = 0 \qquad y^3 - x = 0$$

Substitute $y = x^3$ from EQ1 into EQ2:

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1)$$

$$= x(x^2 - 1)\underbrace{(x^2 + 1)(x^4 + 1)}_{\text{never } 0}$$

The three solutions are $x = 0$, $x = 1$, $x = -1$. Then the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$. (using $y = x^3$)

② Classify the critical points:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$\Rightarrow D = D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

$D(0,0) = -16 < 0 \Rightarrow$ Saddle point

$D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0 \Rightarrow$ Local minimum

$D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0 \Rightarrow$ Local minimum

Example. Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

① Find the critical points

$f_x = 20xy - 10x - 4x^3$ $\qquad$ $f_y = 10x^2 - 8y - 8y^3$

We need to solve the system:

(1) $2x(10y - 5 - 2x^2) = 0$

(2) $5x^2 - 4y - 4y^3 = 0$

From EQ1, $x = 0$ or $x^2 = 5y - 2.5$

If $x = 0$, EQ2 becomes $-4y - 4y^3 = 0 \Rightarrow -4y(1+y^2) = 0 \Rightarrow y = 0$

If $x^2 = 5y - 2.5$, EQ2 becomes $25y - 12.5 - 4y - 4y^3 = 0$

Calculator: $y = -2.55$, $y = 0.65$, $y = 1.90$

use $x^2 = 5y - 2.5$ to get the x-coords ($y = -2.55 \rightarrow$ no solution)

Critical Points: $(0,0)$, $(\pm 2.64, 1.90)$, $(\pm 0.86, 0.65)$

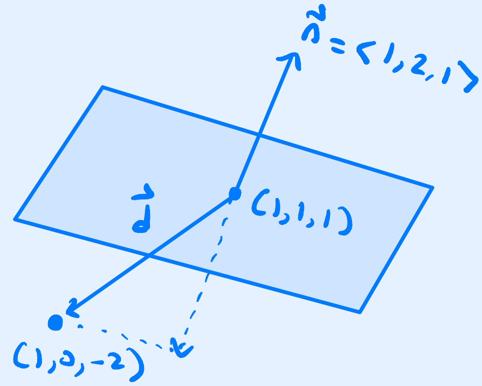
② Classify the critical points:

Critical points	D	f_{xx}	Conclusion
$(0,0)$	80.00	-10.00	Local maximum
$(\pm 2.64, 1.90)$	2488.72	-55.93	Local maximum
$(\pm 0.86, 0.65)$	-187.64	-5.87	Saddle Point

Example. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

Alternative: Compute the absolute value of the scalar projection of $\vec{d}$ onto $\vec{n}$

$$\left|\frac{\vec{d}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{\langle 0,-1,-3\rangle\cdot\langle 1,2,1\rangle}{\sqrt{6}}\right| = \frac{5}{\sqrt{6}}$$



① Find a function $f(x,y)$ to minimize:

The distance between (x,y,z) and $(1,0,-2)$ is

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$$

If (x,y,z) is on the plane, then $z = 4 - x - 2y$. So

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$$

Trick: minimize d^2 instead

$$d^2 = f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2$$

② Find the critical points

Solve the system

(1) $f_x = 4x + 4y - 14 = 0$

(2) $f_y = 4x + 10y - 24 = 0$

$\Rightarrow$ The only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$

③ Classify $\left(\frac{11}{6}, \frac{5}{3}\right)$:

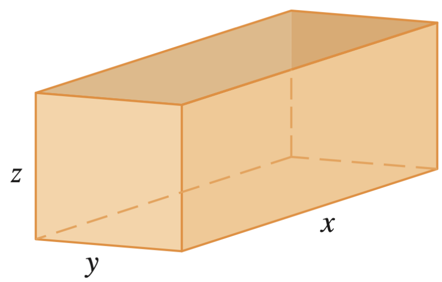
$f_{xx} = 4 \qquad f_{xy} = 4 \qquad f_{yy} = 10$

Point	D	f_{xx}	Conclusion
$\left(\frac{11}{6}, \frac{5}{3}\right)$	24	4	Local Minimum

(Rmk: this is also an absolute minimum ... why?)

$$d = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(6 - \frac{11}{6} - \frac{10}{3}\right)^2} = \frac{5}{\sqrt{6}}$$

Example. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.



① Find a function $f(x,y)$ to maximize.

$$V = xyz$$

Need a function of two variables

Know: $xy + 2xz + 2yz = 12$

$$\Rightarrow V(x,y) = xy\left[\underbrace{\frac{12-xy}{2(x+y)}}_{z}\right] = \frac{12xy - x^2y^2}{2(x+y)}$$

② Find the critical points:

$$\frac{\partial V}{\partial x} = \frac{y^2(12-2xy-x^2)}{2(x+y)^2} \qquad \frac{\partial V}{\partial y} = \frac{x^2(12-2xy-y^2)}{2(x+y)^2}$$

Note: $x>0$ and $y>0$, so these are 0 when

$$12-2xy-x^2=0 \quad \text{and} \quad 12-2xy-y^2=0$$

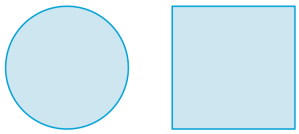
Hence $x^2=y^2 \Rightarrow x=y$.

Substituting back, $12-2x^2-x^2=0$

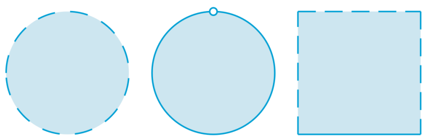
Conclude: $x=2$, $y=2$, $z = \dfrac{12-2\cdot 2}{2(2+2)} = 1$

③ By the Second Derivatives Test, $V(2,2) = 4\,\text{m}^3$ is a local maximum. It is also an absolute maximum (why?)

Definition. What is a closed set? What is a bounded set?



(a) Closed sets



(b) Sets that are not closed

Boundary Point of a set D in $\mathbb{R}^2$:

a point (a,b) such that every disk with center (a,b) contains points in D and also points not in D.

Closed set in $\mathbb{R}^2$: A set that contains all of its boundary points

Bounded set in $\mathbb{R}^2$: A set that is contained in some disk.

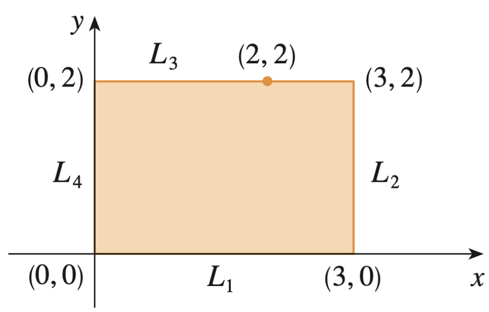
Theorem (Extreme Value Theorem)**.** For a function f of one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. What is the Extreme Value Theorem in two dimensions?

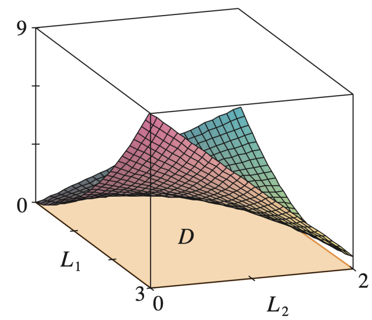
- If $f(x,y)$ is continuous on a closed, bounded set D in $\mathbb{R}^2$, then $f(x,y)$ attains an absolute maximum and an absolute minimum on D.

To find the absolute maximum | minimum:

1. Find the values of f at the critical points of f in D
2. Find the extreme values of f on the boundary of D
3. The largest of these values is the absolute maximum
   The smallest of these values is the absolute minimum

Example. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$.





- f is a polynomial, so it is continuous on the closed, bounded rectangle D, and the E.V.T. applies.

① Solve $f_x = 2x - 2y = 0$ and $f_y = -2x + 2 = 0$

$\Rightarrow$ only critical point is $(1,1)$

$\Rightarrow$ $f(1,1) = 1$

② The boundary consists of the lines L_1, L_2, L_3, L_4

- On L_1, $y = 0$ $\Rightarrow$ $f(x,0) = x^2$, $0 \le x \le 3$
  - increasing function of x
  - minimum is $f(0,0) = 0$
  - maximum is $f(3,0) = 9$
- On L_2, $x = 3$ $\Rightarrow$ $f(3,y) = 9 - 4y$, $0 \le y \le 2$
  - decreasing function of y
  - minimum is $f(3,2) = 1$
  - maximum is $f(3,0) = 9$

- On L_3, $y=2 \Rightarrow f(x,2) = x^2-4x+4$, $0 \le x \le 3$
  - Use methods from calc 1
  - minimum is $f(2,2) = 0$
  - maximum is $f(0,2) = 4$

- On L_4, $x=0 \Rightarrow f(0,y) = 2y$, $0 \le y \le 2$
  - minimum is $f(0,0) = 0$
  - maximum is $f(0,2) = 4$

- Hence, on the boundary:
  - minimum is 0
  - maximum is 9

(3) Absolute minimum is $f(0,0) = f(2,2) = 0$

Absolute maximum is $f(3,0) = 9$