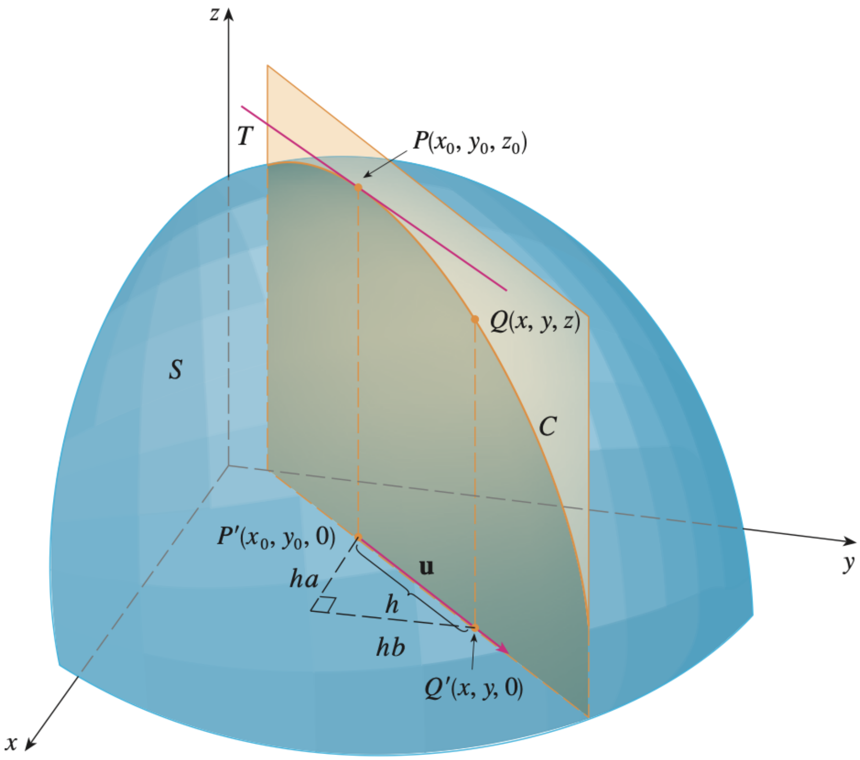
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: ____________________

## 11.6 Directional Derivatives and the Gradient Vector

**Question.** What is the idea of a directional derivative?

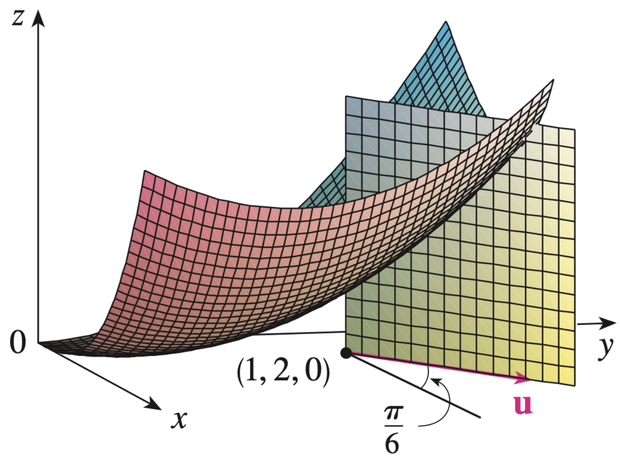


**Definition.** What is the limit definition of the directional derivative of $f$ at $(x_0, y_0)$ in the direction of a unit vector $\vec{u} = \langle a, b \rangle$?

**Remark.** Explain how the partial derivatives $f_x$ and $f_y$ are special cases of the directional derivative.

**Theorem.** If $f$ is a differentiable function of $x$ and $y$, how to compute the directional derivative in the direction of a unit vector $\vec{u} = \langle a, b \rangle$?

**Example.** Find the directional derivative $D_u f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\vec{u}$ is the unit vector given by the angle $\theta = \pi/6$. What is $D_u f(1, 2)$?



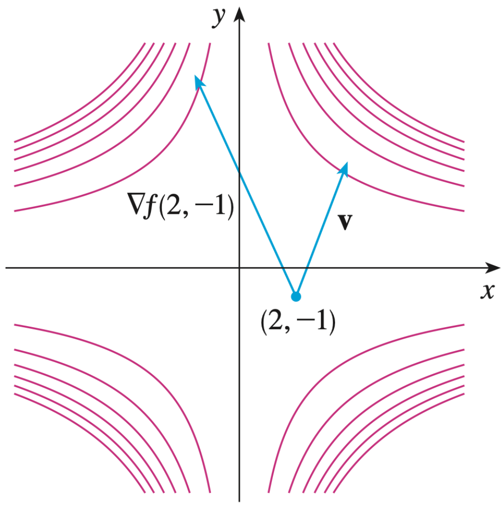

**Question.** How can the directional derivative of a differentiable function be written as the dot product of two vectors?

**Definition.** If $f$ is a function of two variables $x$ and $y$, what is the gradient of $f$?

**Example.** Find $\nabla f$ if $f(x, y) = \sin x + e^{xy}$.

**Remark.** Rewrite the expression for the directional derivative of a differentiable function using $\nabla f$.

**Example.** Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 5\vec{j}$.




**Definition.** For a function $f(x, y, z)$ of three variables, what is the limit definition of the directional derivative of $f$ at $(x_0, y_0, z_0)$ in the direction of a unit vector $\vec{u} = \langle a, b, c \rangle$?

**Definition.** For a function $f(x, y, z)$ of three variables, how to compute the directional derivative of $f$ at $(x_0, y_0, z_0)$ in the direction of a unit vector $\vec{u} = \langle a, b, c \rangle$?

**Definition.** For a function $f(x, y, z)$ of three variables, what is the gradient vector of $f$? Rewrite the formula for the directional derivative of using the gradient vector.

**Example.** Let $f(x, y, z) = x \sin yz$.

(a) Find the gradient of $f$

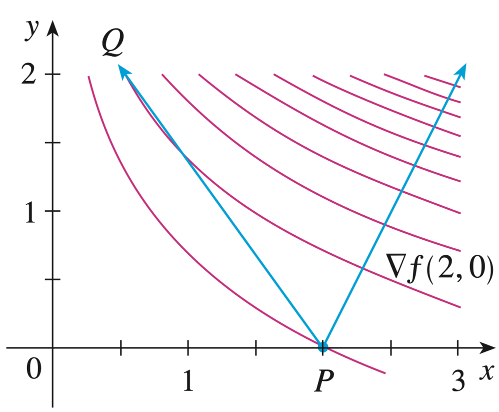
(b) Find the directional derivative of $f$ at $(1, 3, 0)$ in the direction of $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$.

**Theorem.** Suppose we have a function $f$ of two or three variables and we consider all possible directional derivatives of $f$ at a given point. In which of these directions does $f$ change fastest and what is the maximum rate of change?

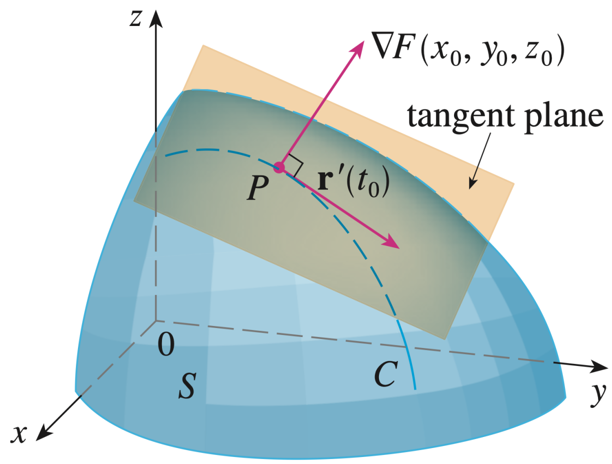
*Proof.*

$\square$

**Example.** If $f(x, y) = xe^y$, find the rate of change of $f$ at the point $P(2, 0)$ in the direction from $P$ to $Q(\frac{1}{2}, 2)$. In what direction does $f$ have the maximum rate of change? What is this maximum rate of change?

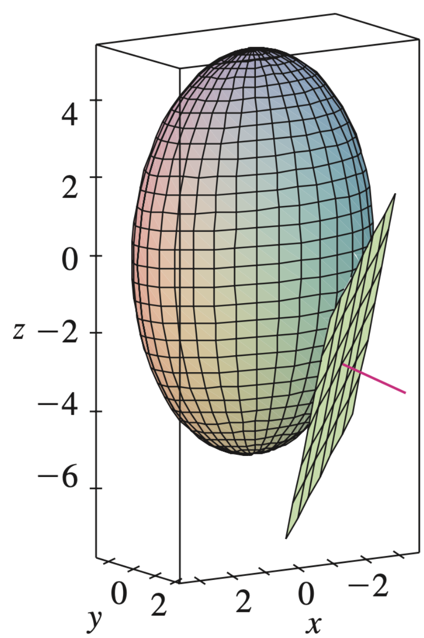


**Example.** Suppose $S$ is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function $F$ of three variables, and let $P(x_0, y_0, z_0)$ be a point on $S$. What is an equation of the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$?

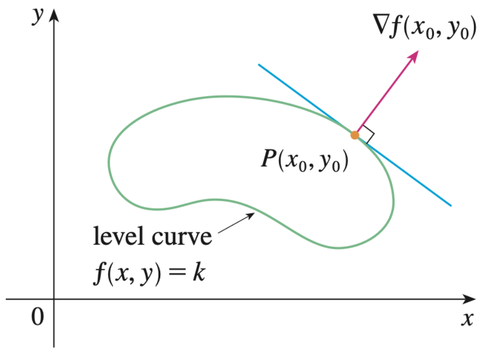


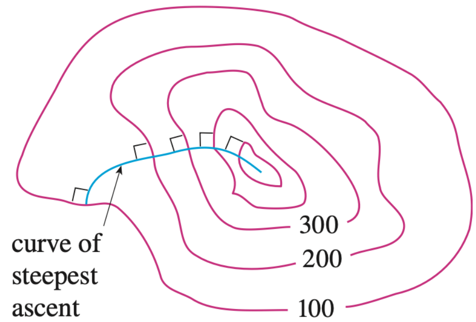
**Question.** How is the equation of a tangent plane to a surface $S$ that is the graph of a function $f$ of two variables a special case of the above?

**Example.** Find the equation of the tangent plane at the point $(2, 1, 3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.



**Question.** For a function $f$ of two variables, explain how the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of $f$.




**Example.** The picture below is a contour map of the function $f(x, y) = x^2 - y^2$. Gradient vectors at various points have also been plotted. This is called a gradient vector field.