Lecture Notes
Math 2400 - Calculus III
Spring 2024

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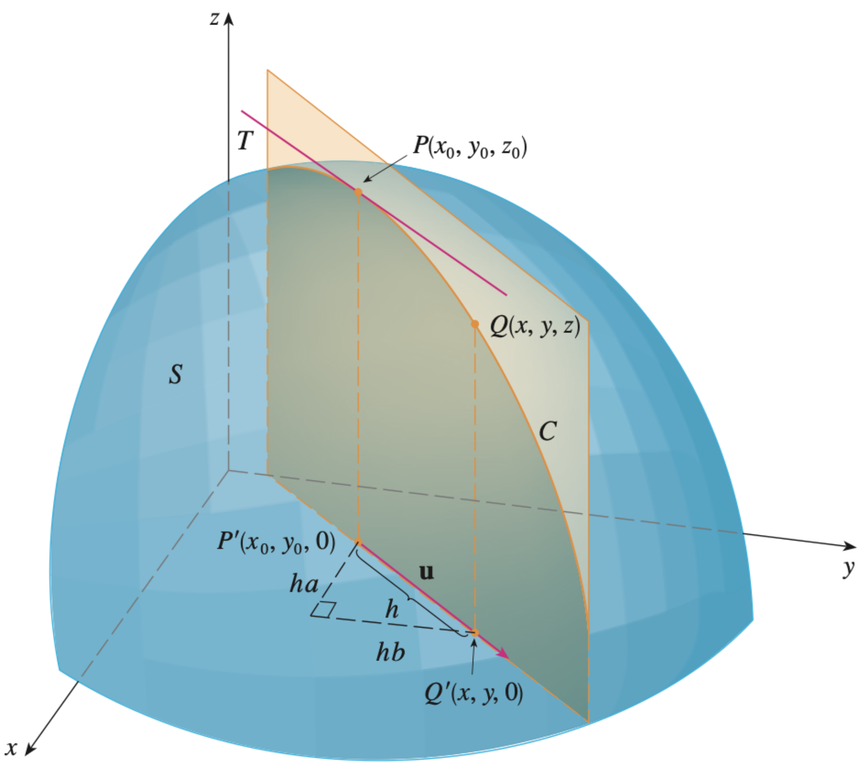
11.6 Directional Derivatives and the Gradient Vector

Question. What is the idea of a directional derivative?



- Given an arbitrary unit vector $\vec{u} = \langle a, b \rangle$, the vertical plane in the direction of $\vec{u}$ intersects S in a curve C.

- The slope of the tangent line T to the curve C at the point P is the derivative of $z = f(x,y)$ in the direction of $\vec{u}$.

Definition. What is the limit definition of the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$?

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Remark. Explain how the partial derivatives f_x and f_y are special cases of the directional derivative.

If we take $\vec{u} = \vec{i} = \langle 1, 0 \rangle$, then $D_{\vec{i}} f = f_x$

If we take $\vec{u} = \vec{j} = \langle 0, 1 \rangle$, then $D_{\vec{j}} f = f_y$

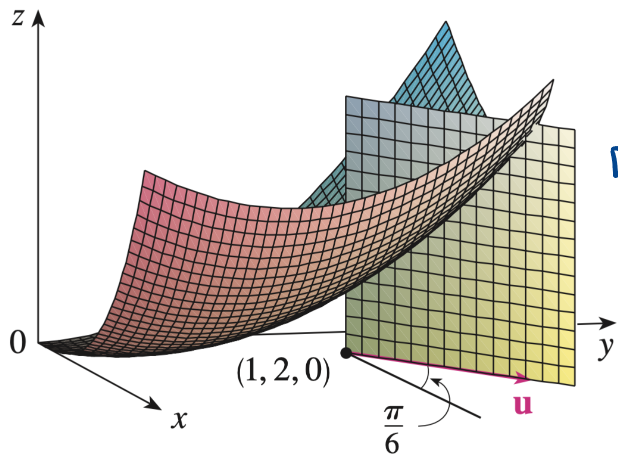
Theorem. If f is a differentiable function of x and y, how to compute the directional derivative in the direction of a unit vector $\vec{u} = \langle a, b \rangle$?

$$D_u f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$$

If $\vec{u}$ makes angle θ with the positive x-axis, then $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ and

$$D_u f(x,y) = f_x(x,y) \cos\theta + f_y(x,y) \sin\theta$$

Example. Find the directional derivative $D_u f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{u}$ is the unit vector given by the angle $\theta = \pi/6$. What is $D_u f(1,2)$?



$$D_u f(x,y) = f_x(x,y)\cos\left(\tfrac{\pi}{6}\right) + f_y(x,y)\sin\left(\tfrac{\pi}{6}\right)$$

$$= (3x^2 - 3y)\cdot\frac{\sqrt{3}}{2} + (-3x+8y)\cdot\frac{1}{2}$$

At the point (1,2),

$$D_u f(1,2) = (3\cdot 1^2 - 3\cdot 2)\frac{\sqrt{3}}{2} + (-3\cdot 1 + 8\cdot 2)\cdot\frac{1}{2} = \frac{13 - 3\sqrt{3}}{2}$$

Question. How can the directional derivative of a differentiable function be written as the dot product of two vectors?

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$$

$$= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle$$

$$= \underbrace{\langle f_x(x,y), f_y(x,y)\rangle}_{} \cdot \vec{u}$$

This is the gradient vector

Definition. If f is a function of two variables x and y, what is the gradient of f?

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

Example. Find ∇f if $f(x, y) = \sin x + e^{xy}$.

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \cos x + y e^{xy}, x e^{xy} \rangle$$

For example, at the point $(0,1)$:

$$\nabla f(0,1) = \langle 2, 0 \rangle$$

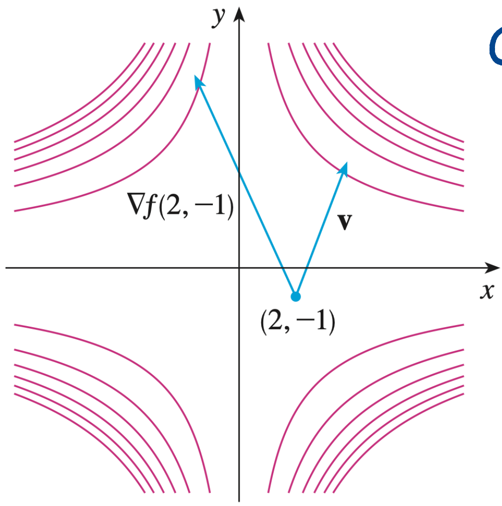
Remark. Rewrite the expression for the directional derivative of a differentiable function using ∇f.

$$D_u f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

Note: this is the scalar projection of ∇f onto $\vec{u}$

$$\text{comp}_{\vec{u}} \nabla f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} = \nabla f \cdot \vec{u}$$

Example. Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 5\vec{j}$.



① Gradient vector:

$$\nabla f(x,y) = \langle 2xy^3,\ 3x^2y^2 - 4 \rangle$$

$$\nabla f(2,-1) = \langle -4,\ 8 \rangle$$

② Since $|\vec{v}| = \sqrt{29}$, the unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \left\langle \frac{2}{\sqrt{29}},\ \frac{5}{\sqrt{29}} \right\rangle$$

③ Therefore,

$$D_u(2,-1) = \nabla f(2,-1) \cdot \vec{u} = \langle -4, 8 \rangle \cdot \left\langle \frac{2}{\sqrt{29}},\ \frac{5}{\sqrt{29}} \right\rangle = \frac{32}{\sqrt{29}}$$

Definition. For a function $f(x, y, z)$ of three variables, what is the limit definition of the directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b, c\rangle$?

$$D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha,\ y_0 + hb,\ z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Definition. For a function $f(x, y, z)$ of three variables, how to compute the directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} = \langle a, b, c \rangle$?

$$D_u f(x,y,z) = f_x(x,y,z) \cdot a + f_y(x,y,z) \cdot b + f_z(x,y,z) \cdot c$$

Definition. For a function $f(x, y, z)$ of three variables, what is the gradient vector of f? Rewrite the formula for the directional derivative of using the gradient vector.

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

Then the formula for the directional derivative becomes

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

Example. Let $f(x, y, z) = x \sin yz$.

(a) Find the gradient of f

(b) Find the directional derivative of f at $(1, 3, 0)$ in the direction of $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$.

(a) The gradient of f is

$$\nabla f(x,y,z) = \langle \sin yz, xz \cos yz, xy \cos yz \rangle$$

(b) At the point $(1,3,0)$, $\nabla f(1,3,0) = \langle 0, 0, 3 \rangle$

The unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$$

$$D_u f(1,3,0) = \nabla f(1,3,0) \cdot \vec{u} = \langle 0,0,3 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right\rangle$$

$$= \frac{-3}{\sqrt{6}}$$

Theorem. Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. In which of these directions does f change fastest and what is the maximum rate of change?

- The maximum value of $D_u f(x,y)$ is $|\nabla f(x,y)|$
- It occurs when $\vec{u}$ has the same direction as $\nabla f(x,y)$

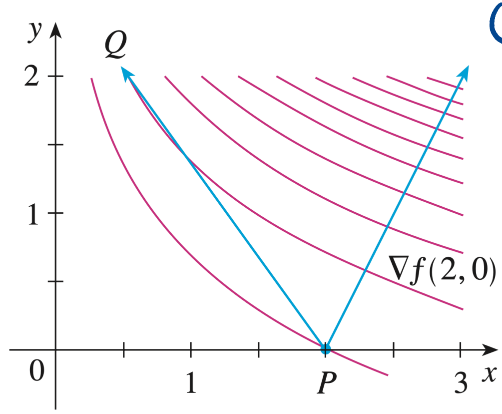
Proof.

$$D_u f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta = |\nabla f|\cos\theta$$

where θ is the angle between ∇f and $\vec{u}$. The maximum value of $\cos\theta$ is 1, when $\theta = 0$. That is, ∇f and $\vec{u}$ have the same direction.

□

Example. If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$. In what direction does f have the maximum rate of change? What is this maximum rate of change?



① Gradient vector:

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$

$\nabla f(2,0) = \langle 1, 2 \rangle$

② The vector $\vec{PQ} = \langle -1.5, 2 \rangle$, so

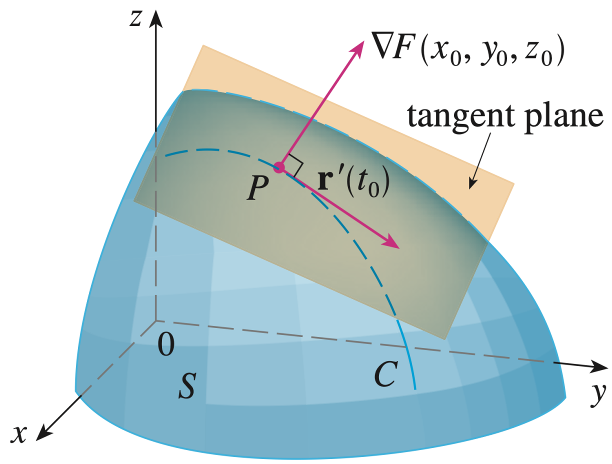
$\vec{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

Hence $D_u f(2,0) = \nabla f(2,0) \cdot \vec{u} = \langle 1,2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = 1$

③ f increases the fastest in the direction of $\nabla f(2,0) = \langle 1,2 \rangle$

The maximum rate of change is $|\nabla f(2,0)| = |\langle 1,2 \rangle| = \sqrt{5}$

Example. Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. What is an equation of the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$?



- Let C be any curve on S that passes through P
- C is described by a vector function

  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

  where, say, $\vec{r}(t_0) = P$

- Since C is on the surface, $F(x(t), y(t), z(t)) = k$
- By the chain rule, $\frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} = 0$
- We can rewrite this as $\nabla F \cdot \vec{r}\,'(t) = 0$
- In particular, when $t = t_0$, this says $\nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$
- Conclude: $\nabla F(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\vec{r}\,'(t_0)$ to any curve passing through P

Use ∇F as the normal vector to the tangent plane.

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

Question. How is the equation of a tangent plane to a surface S that is the graph of a function f of two variables a special case of the above?

If S is the graph of $z = f(x,y)$, we can let $F(x,y,z) = f(x,y) - z$ and consider the level surface $F(x,y,z) = 0$.

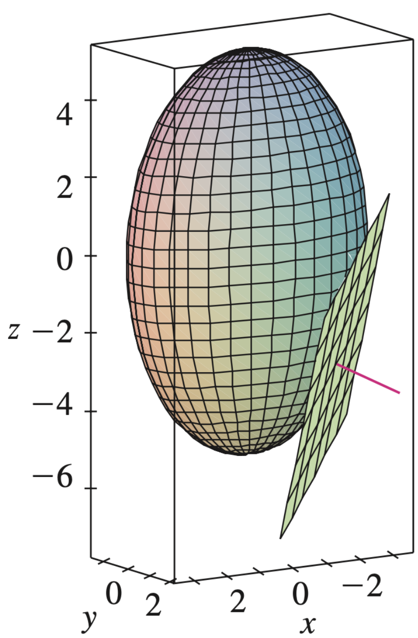
Then
$$F_x(x_0,y_0,z_0) = f_x(x_0,y_0)$$
$$F_y(x_0,y_0,z_0) = f_y(x_0,y_0)$$
$$F_z(x_0,y_0,z_0) = -1$$

In this case, we get

$$\boxed{f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0}$$

Example. Find the equation of the tangent plane at the point [redacted] to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

$(-2,1,-3)$



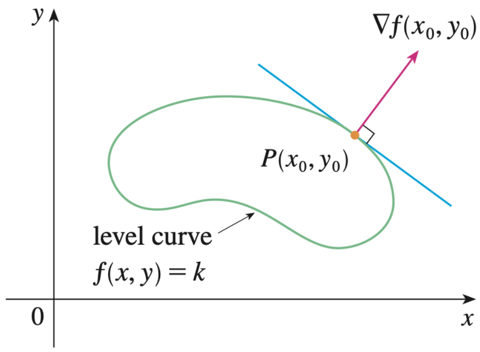
- Let $F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$
- The ellipsoid is the level surface $F(x,y,z) = 3$
- We have

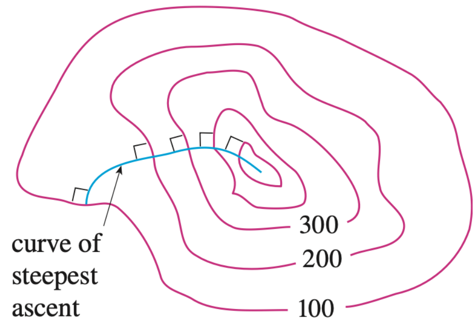
$$F_x(x,y,z) = \frac{x}{2} \Rightarrow F_x(-2,1,-3) = -1$$
$$F_y(x,y,z) = 2y \Rightarrow F_y(-2,1,-3) = 2$$
$$F_z(x,y,z) = \frac{2z}{9} \Rightarrow F_z(-2,1,-3) = -\frac{2}{3}$$

The tangent plane is

$$\boxed{-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0}$$

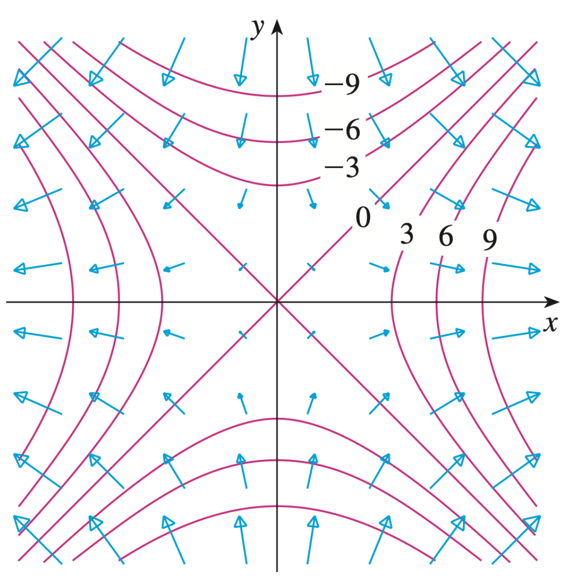
Question. For a function f of two variables, explain how the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f.





- We showed that for a function $F(x,y,z)$, ∇F is orthogonal to any level surface $F(x,y,z) = k$.
- A similar calculation shows for a function $f(x,y)$, ∇f is orthogonal to any level curve $f(x,y) = k$
- This corresponds to ∇f pointing in the direction of fastest increase.

Example. The picture below is a contour map of the function $f(x, y) = x^2 - y^2$. Gradient vectors at various points have also been plotted. This is called a gradient vector field.



The gradient vectors are perpendicular to the level curves.