Lecture Notes
Math 2400 - Calculus III
Spring 2024

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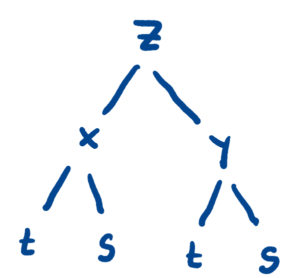
11.5 The Chain Rule

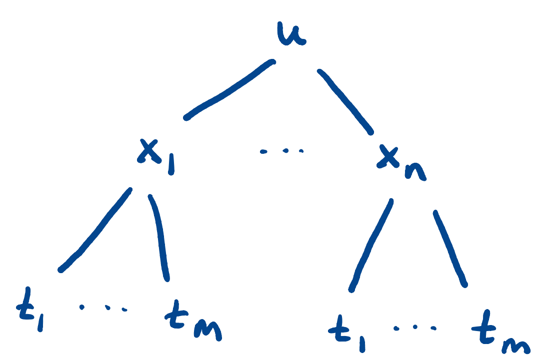
Question. For functions of more than one variable, the chain rule has several versions. What are the three different cases that we will be looking at?

1. z is a function of x and y and x and y are both functions of t.

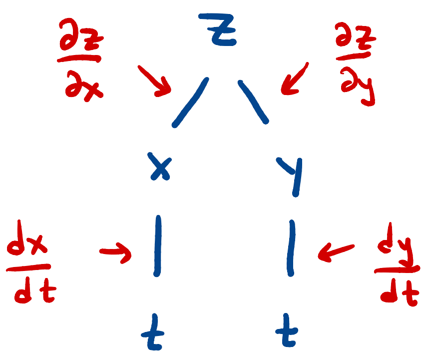
2. z is a function of x and y and x and y are functions of t and s.



3. u is a function of $x_1, \dots, x_n$ and each x_j is a function of $t_1, \dots, t_m$



Definition (Chain Rule: Case 1)**.** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. What is the chain rule in this case?



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Example. If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}$ when $t = 0$.

The chain rule says

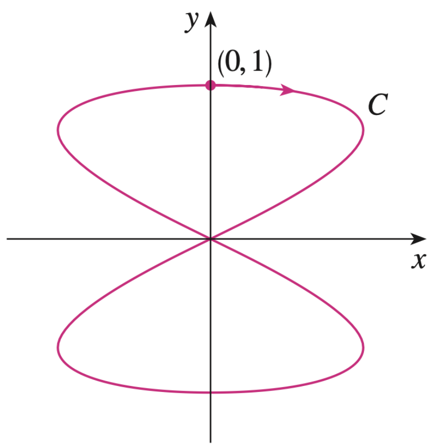
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy + 3y^4) \cdot (2\cos 2t) + (x^2 + 12xy^3) \cdot (-\sin t)$$

When $t = 0$, $x = 0$ and $y = 1$

$$\frac{dz}{dt} = (0 + 3) \cdot (2) + (0 + 0) \cdot (0) = 6$$

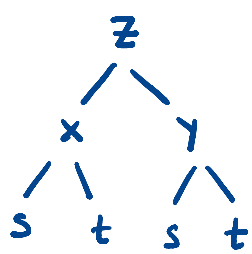
Question. In the above example, suppose that $z = T(x, y) = x^2y + 3xy^4$ represents the temperature at the point (x, y). What does $\frac{dz}{dt}$ mean?



$\frac{dz}{dt}$ is the rate at which the temperature changes as we move along the curve

The rate of increase is 6 as move through the point (0,1)

Definition (Chain Rule: Case 2)**.** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. What is the chain rule in this case?



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

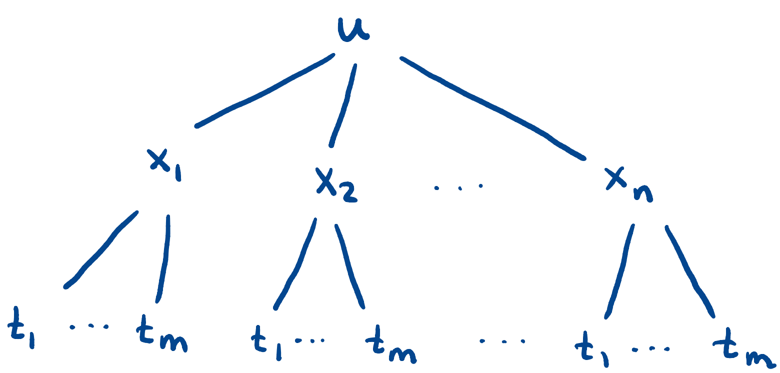
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Example. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = (e^x \sin y) \cdot (t^2) + (e^x \cos y) \cdot (2st)$$

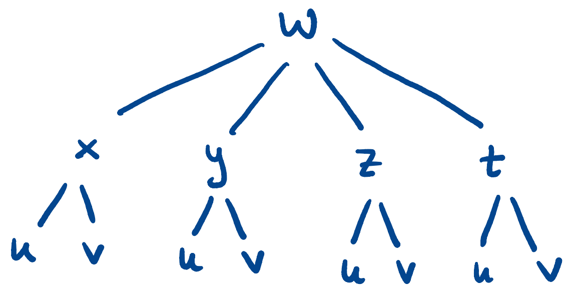
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = (e^x \sin y) \cdot (2st) + (e^x \cos y) \cdot (s^2)$$

Definition (Chain Rule: General Version)**.** Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. What is the chain rule in this case?



$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

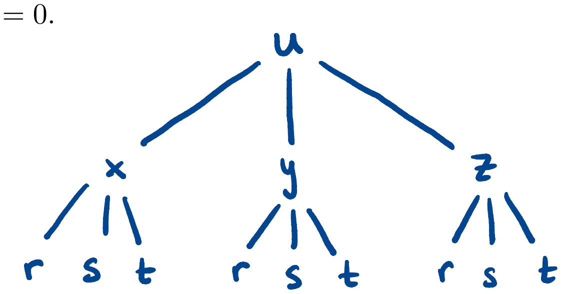
Example. Write out the chain rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.



$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial v}$$

Example. If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\dfrac{\partial u}{\partial s}$ when $r = 2, s = 1, t = 0$.



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= (4x^3y)\cdot(re^t) + (x^4 + 2yz^3)\cdot(2rse^{-t}) + (3y^2z^2)\cdot(r^2\sin t)$$

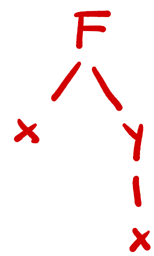
When $r=2$, $s=1$, $t=0$ ... $x = 2$, $y = 2$, $z = 0$. So,

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$

Definition. Suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x. How can we use the chain rule to solve for $\frac{dy}{dx}$?

Use the chain rule to differentiate $F(x,y) = 0$

$$\Rightarrow \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$



If $\partial F / \partial y \neq 0$, we get

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{-F_x}{F_y}$$

Example. Find y' if $x^3 + y^3 = 6xy$.

$$\xrightarrow{\text{Calc 1}} \quad 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6\left[x \cdot \frac{dy}{dx} + y \cdot 1\right]$$

Solve for $\frac{dy}{dx}$

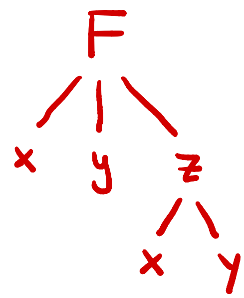
Define $F(x,y) = x^3 + y^3 - 6xy$

Then

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Definition. Suppose that an equation of the form $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y. How can we use the chain rule to solve for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$?

Use the chain rule to differentiate $F(x, y, z) = 0$ w.r.t. x and y



1. $$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

2. $$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} \quad \text{AND} \quad \frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}}$$

Example. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Note: We did this computation in §11.3