Lecture Notes
Math 2400 - Calculus III
Spring 2024

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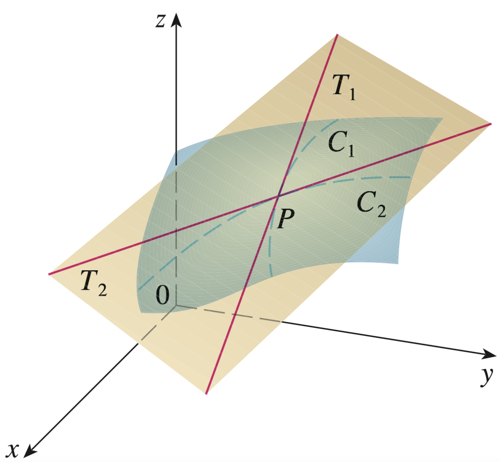
11.4 Tangent Planes and Linear Approximations

Question. Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives. What is the tangent plane to the surface S at the point P?



- let $P(x_0, y_0, z_0)$ be a point on the surface S
- the planes $x = x_0$ and $y = y_0$ intersect S in the curves C_1 and C_2
- T_1 and T_2 are the tangent lines to the curves C_1 and C_2 at the point P
- The tangent plane is the plane that contains both T_1 and T_2.

Fact: If C is any other curve on S that passes through P, its tangent line at P also lies in the tangent plane.

Question. Suppose f has continuous partial derivatives. What is an equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$? Relate this to the equation of a tangent line of a function $f(x)$.

Equation of tangent plane to $f(x,y)$:

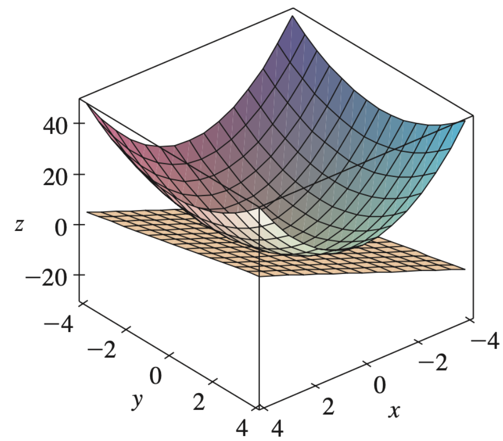
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note: if we intersect the tangent plane with the plane $y = y_0$, we obtain a line with slope f_x

Equation of tangent line to $f(x)$:

$$y - y_0 = f'(x_0)(x - x_0)$$

Example. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.



$z - z_0 = f_x \cdot (x - x_0) + f_y \cdot (y - y_0)$

$(x_0, y_0, z_0) = (1, 1, 3)$

$f_x = 4x$ $\qquad$ $f_y = 2y$

$f_x(1,1) = 4$ $\qquad$ $f_y(1,1) = 2$

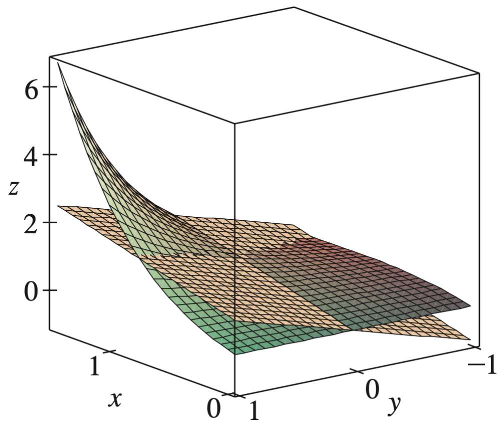
$$\boxed{z - 3 = 4(x-1) + 2(y-1)}$$

OR: $\quad z = 4x + 2y - 3$

Definition. What is the linearization of $f(x, y)$ at (a, b)? What is the the linear approximation of $f(x, y)$ at (a, b)?

- Idea: Near (a,b), the tangent plane is a reasonable approximation of the graph of $f(x,y)$
- Linearization: The equation $L(x,y)$ of the tangent plane
  - $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
  - In the above example, $L(x,y) = 4x + 2y - 3$
- Linear approximation: The approximation $f(x,y) \approx L(x,y)$

Example. Find the linearization of $f(x,y) = xe^{xy}$ at $(1,0)$. Then use it to approximate $f(1.1,-0.1)$.



① Find f_x and f_y :

$$f_x = e^{xy} + xye^{xy}$$

$$f_x(1,0) = 1$$

$$f_y = x^2 e^{xy}$$

$$f_y(1,0) = 1$$

② Find the linearization:

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

$$L(x,y) = 1 + 1(x-1) + 1(y-0)$$

$$L(x,y) = x + y$$

Near the point $(1,0)$ $f(x,y)$ is approximately $x+y$

③ $f(x,y) \approx L(x,y) = x + y$

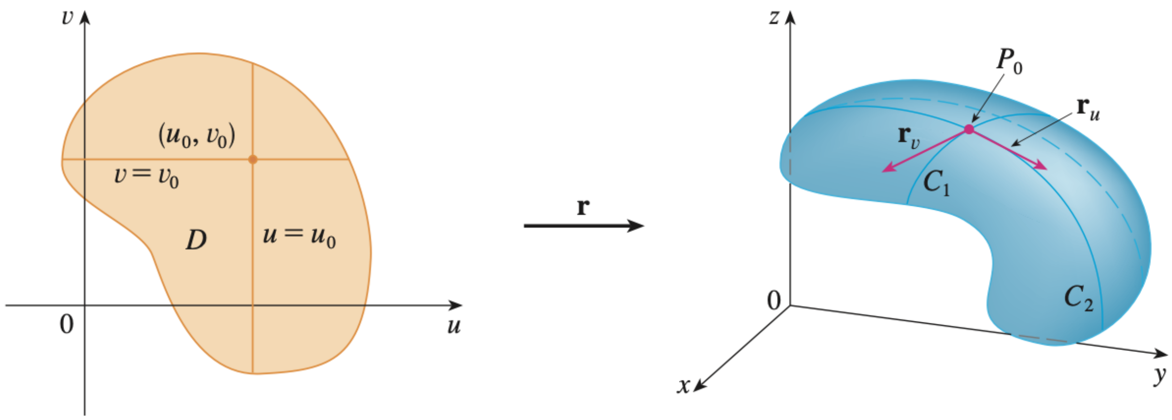
$$f(1.1,-0.1) \approx 1.1 - 0.1 = 1$$

Check: $f(1.1,-0.1) = 1.1e^{-0.11} = 0.98542$

Example. How can we find the tangent plane to a parametric surface S traced out by a vector function

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

at a point P_0 with position vector $\vec{r}(u_0, v_0)$?



- If we fix $u = u_0$, we get the grid curve C_1 on the surface
- The tangent vector to C_1 at P is

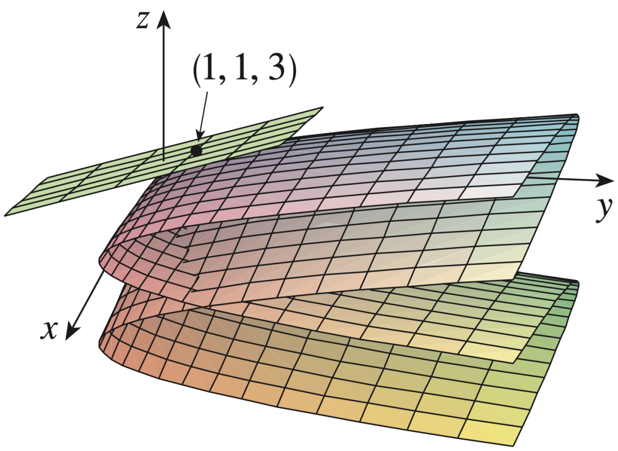
$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\vec{k}$$

- Similarly, fixing $v = v_0$ gives the grid curve C_2, given by $\vec{r}(u, v_0)$. Its tangent vector at P is

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\vec{k}$$

- The tangent plane is the plane containing the vectors $\vec{r}_u$ and $\vec{r}_v$ and $\vec{r}_u \times \vec{r}_v$ is a normal vector to the plane.

Example. Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.



① The vector function is

$$\vec{r}(u,v) = \langle u^2, v^2, u+2v \rangle$$

② Tangent vectors:

$$\vec{r}_u = \langle 2u, 0, 1 \rangle$$

$$\vec{r}_v = \langle 0, 2v, 2 \rangle$$

③ Normal vector

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\,\vec{i} - 4u\,\vec{j} + 4uv\,\vec{k}$$

$$= \langle -2v, -4u, 4uv \rangle$$

The point $(1,1,3)$ corresponds to $u=1$ and $v=1$,
so the normal vector at $(1,1,3)$ is $\langle -2, -4, 4 \rangle$

④ The equation of the tangent plane at $(1,1,3)$ is

$$-2(x-1) - 4(y-1) + 4(z-3) = 0$$

$$\iff \quad x + 2y - 2z + 3 = 0$$