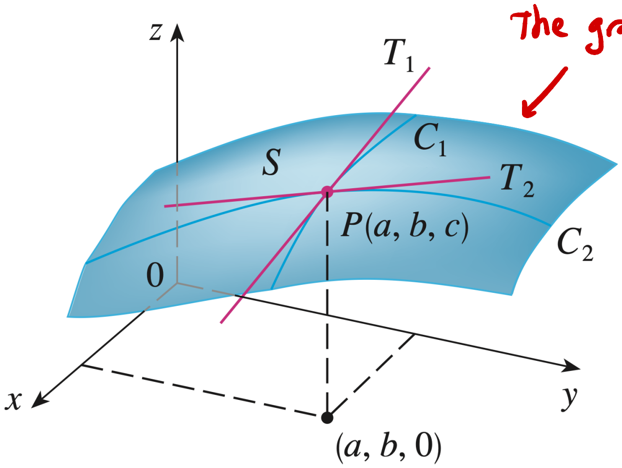
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: ____________________

11.3 Partial Derivatives

Question. What is a partial derivative geometrically?



The graph of $z = f(x,y)$

- At a point P, slice the surface $z = f(x,y)$ parallel to the x-axis and the y-axis
- Obtain curves C_1 and C_2

- The partial derivatives are the slopes of the tangent lines at P to C_1 and C_2 in the planes $y=b$ and $x=a$

↳ The slopes of T_1 and T_2

Definition. What are the partial derivatives of $f(x, y)$?

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

The partial derivative in the x direction captures how much $f(x,y)$ is changing as we fix y and vary x

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Remark. How to compute the partial derivatives of $z = f(x, y)$?

To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Remark. If $z = f(x, y)$, what are various notations for the partial derivatives?

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y}$$

Example. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

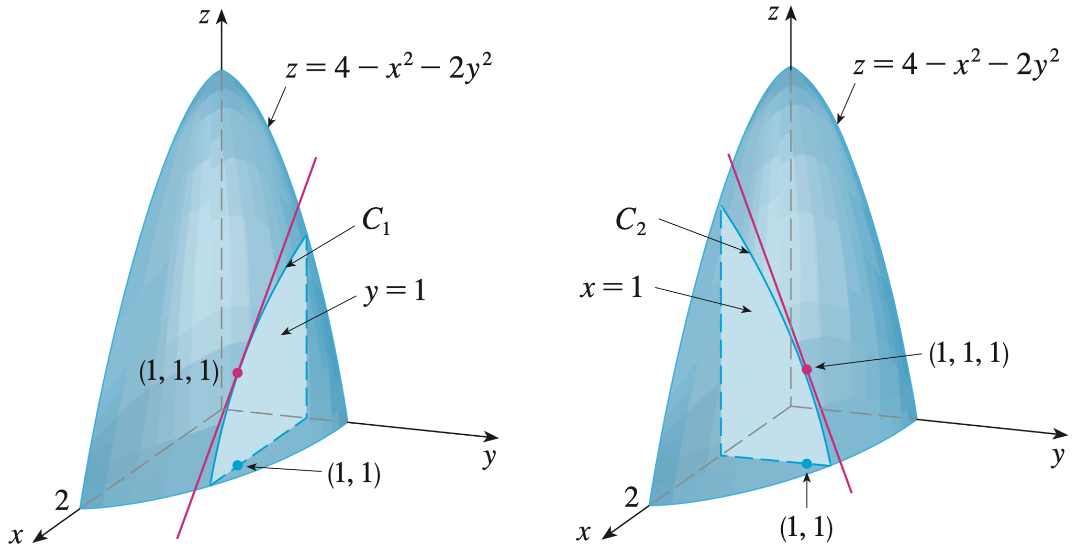
$$f_x(x,y) = 3x^2 + 2xy^3$$

$$f_x(2,1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Example. If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.



$f_x(x,y) = -2x$ $\qquad$ $f_y(x,y) = -4y$

$f_x(1,1) = -2$ $\qquad$ $f_y(1,1) = -4$

The vertical plane $y=1$ intersects the graph in the curve $z = 2 - x^2$. The slope of the tangent line to this curve at $x=1$ is -2.

Example. If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

We need to use the chain rule for one variable:

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{-x}{(1+y)^2}$$

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

I will find $\frac{\partial z}{\partial x}$:

product rule

$$3x^2 + 0 + 3z^2 \cdot \frac{\partial z}{\partial x} + 6y\left[x\frac{\partial z}{\partial x} + z \cdot 1\right] = 0$$

$$3x^2 + 6yz = -3z^2\frac{\partial z}{\partial x} - 6xy\frac{\partial z}{\partial x}$$

$$x^2 + 2yz = \frac{\partial z}{\partial x} \cdot (-z^2 - 2xy)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

Definition. In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, what is the partial derivative with respect to the ith variable x_i?

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

Example. Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy}\ln(z)$.

$$f_x(x,y,z) = y e^{xy} \ln(z)$$

$$f_y(x,y,z) = x e^{xy} \ln(z)$$

$$f_z(x,y,z) = \frac{e^{xy}}{z}$$

Definition. What are the second partial derivatives of $z = f(x, y)$?

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \cdots$$

$$(f_y)_y = f_{yy} = \cdots$$

Example. Find the second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$.

We have calculated

$$f_x = 3x^2 + 2xy^3 \qquad f_y = 3x^2y^2 - 4y$$

$$f_{xx} = 6x + 2y^3 \qquad f_{yx} = 6xy^2$$

$$f_{xy} = 6xy^2 \qquad f_{yy} = 6x^2y - 4$$

Theorem. What does Clairaut's Theorem say about mixed partials?

- If f_{xy} and f_{yx} are both continuous on a disk D that contains (a, b) then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

- We can also define partial derivatives of order 3 or higher:

$$f_{xyy} = (f_{xy})_y = \frac{\partial^3 f}{\partial y^2 \partial x}$$

Clairaut's theorem shows that $f_{xyy} = f_{yxy} = f_{yyx}$ if the functions are continuous

Definition. What is Laplace's equation? What are solutions to this equation called?

- Laplace's Equation: $u_{xx} + u_{yy} = 0$
- Solutions are called harmonic functions. Show up in heat conduction and fluid flow

In P.D.E's you will find solutions to equations like these

Example. Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

$u_x = e^x \sin y$ $\qquad$ $u_y = e^x \cos y$

$u_{xx} = e^x \sin y$ $\qquad$ $u_{yy} = -e^x \sin y$

$\Rightarrow \quad u_{xx} + u_{yy} = 0 \quad \checkmark$

Definition. What is the wave equation? What does this describe?

Wave equation: $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$

This describes the motion of a waveform:

ocean wave, sound wave, light wave ...

Example. Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.

$u_x = \cos(x - at)$ $\qquad$ $u_t = -a\cos(x-at)$

$u_{xx} = -\sin(x-at)$ $\qquad$ $u_{tt} = -a^2 \sin(x-at)$

$\Rightarrow \quad u_{tt} = a^2 u_{xx}$