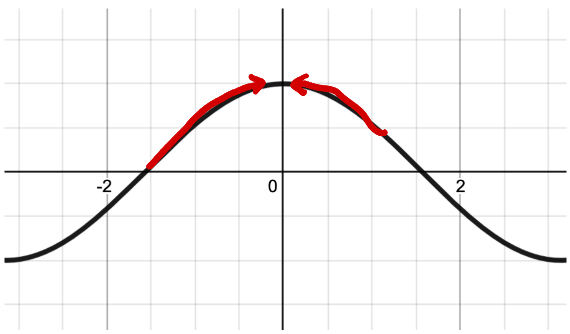
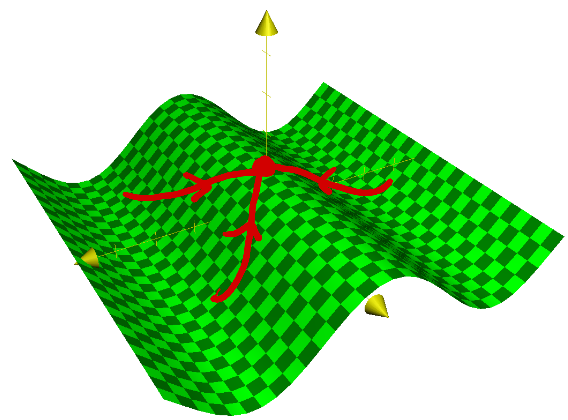
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: ____________________________

11.2 Limits and Continuity

Question. How is evaluating limits of functions in three dimensions different from evaluating limits in two dimensions?





- In 2D, for $\lim_{x \to a} f(x) = L$, we need to check

$\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

- In 3D, for $\lim_{(x,y) \to (a,b)} f(x,y) = L$, we need to check

every possible path approaching (a,b)

Definition. What does it mean for the limit of $f(x, y)$ as (x, y) approaches (a, b) to be L?

$\lim_{(x,y) \to (a,b)} f(x,y) = L$ if we can make the values of

$f(x,y)$ as close to L as we like by taking

(x,y) sufficiently close to (a,b), but not

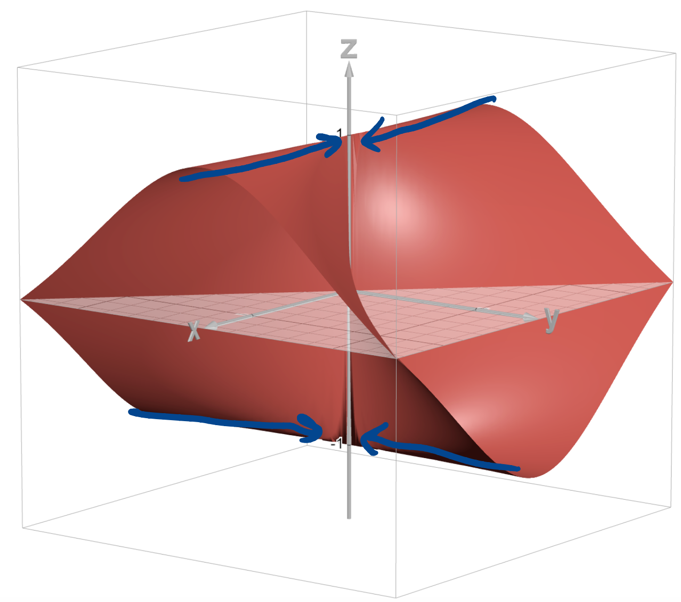
equal to (a,b).

Question. How can we easily show that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist?

- Showing a limit exists is hard, because we must check infinitely many paths.

- To show a limit does not exist, can find two paths along which $f(x,y)$ approaches different values.

Example. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.



① Approach $(0,0)$ along the x-axis:

In this case, $y=0$, so $f(x,0) = \dfrac{x^2-0}{x^2+0^2} = 1$

Hence $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the x-axis.

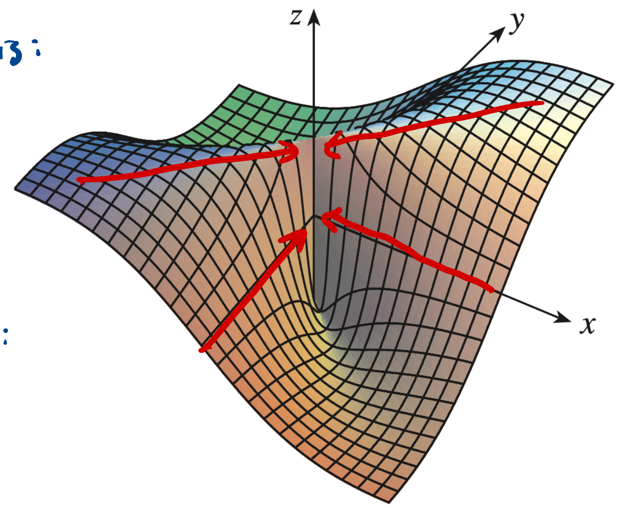
② Approach $(0,0)$ along the y-axis. In this case, $x=0$, so $f(0,y) = \dfrac{0-y^2}{0+y^2} = -1$. Hence $f(x,y) \to -1$ as $(x,y) \to (0,0)$ along the y-axis.

Conclude: The limit does not exist.

Example. Does $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

① Let $(x,y) \to (0,0)$ along x-axis:

$$f(x,0) = \frac{x \cdot 0}{x^2 + 0^2} = 0$$



② Let $(x,y) \to (0,0)$ along y-axis:

$$f(0,y) = \frac{0 \cdot y}{0^2 + y^2} = 0$$

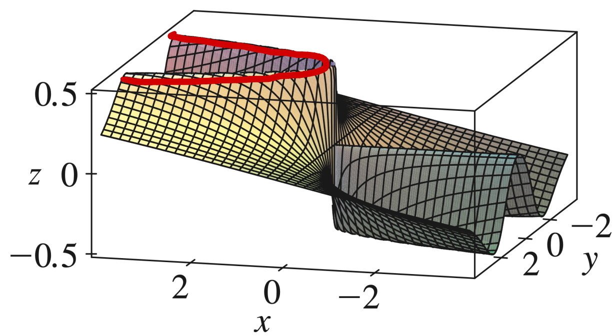
This is not enough to conclude $f(x,y) \to 0$.

③ Let $(x,y) \to (0,0)$ along $y = x$

$$f(x,y) = \frac{x \cdot x}{x^2 + x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

Conclude: Since we have obtained different limits along different paths, the limit does not exist.

Example. Does $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ exist?



- To save time, let $(x,y) \to (0,0)$ along any nonvertical line $y = mx$.
- $f(x,y) = f(x, mx) = \dfrac{x \cdot m^2x^2}{x^2 + m^4x^4} = \dfrac{m^2x^3}{x^2+m^4x^4} = \dfrac{m^2x}{1+m^4x^2}$
- Hence $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = mx$
- This is not enough to show the limit exists.
- Let $(x,y) \to (0,0)$ along the parabola $x = y^2$
- $f(x,y) = f(y^2, y) = \dfrac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \dfrac{y^4}{2y^4} = \dfrac{1}{2}$
- Hence $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $x = y^2$

Conclude: the limit does not exist.

Example. Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

- We suspect the limit is 0:

$$f(x,0) = \frac{0}{x^2} = 0$$

it is 0 along the x-axis

- To prove a limit exists, we must use the Squeeze Theorem

- The difference between $f(x,y)$ and 0 is

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| = \left| \frac{3x^2y}{x^2+y^2} \right| = \frac{3x^2|y|}{x^2+y^2}$$

- Since $y^2 \geq 0$, we know $x^2+y^2 \geq x^2$. Hence $\frac{x^2}{x^2+y^2} \leq 1$.

$$\Rightarrow \quad 0 \leq \frac{3x^2|y|}{x^2+y^2} \leq 3|y|$$

- Since $\lim_{(x,y)\to(0,0)} 0 = 0$ and $\lim_{(x,y)\to(0,0)} 3|y| = 0$, we conclude $\lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} = 0$ as well by the Squeeze thm.

- We've shown the difference between $f(x,y)$ and 0 approaches 0 as $(x,y)\to(0,0)$. Hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Definition. What does it mean for a function $f(x, y)$ to be continuous at (a, b). How to evaluate limits of continuous functions?

- $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

- We can evaluate limits of continuous functions by direct substitution.

Example. What are some examples of continuous functions?

- Sums, differences, products, and quotients of continuous functions are continuous on their domains

- Polynomials and rational functions are continuous on their domains

A ratio of polynomials

Sum of terms $cx^m y^n$ e.g $f(x,y) = x^4 + 5x^3y^2 - 7y + 6$

Example. Evaluate $\lim_{(x,y)\to(1,2)} x^2y^2 - x^3y^2 + 3x + 2y$.

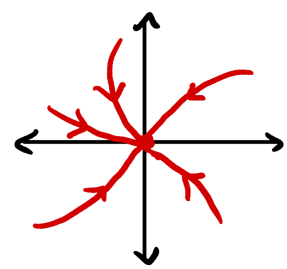
- This is a polynomial, so it is continuous everywhere

- The limit is $1^2 \cdot 2^2 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 7$

Example. Where is the function $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ continuous?

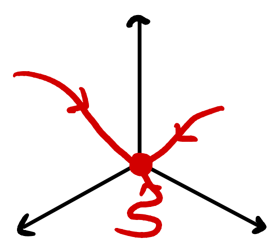
- This is a rational function, so it is continuous on its domain
- Its domain is the set $D = \{(x,y) \mid (x,y) \neq (0,0)\}$

Example. How can we extend things to functions of three or more variables?



$f(x,y)$

- $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$ if $f(x,y,z) \to L$ as $(x,y,z) \to (a,b,c)$ along any path



$f(x,y,z)$

- f is continuous at (a,b,c) if $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$

Example. Where is the function $f(x,y,z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$ continuous?

- $\sqrt{y}$ is continuous for $y \geq 0$
- Also need $x^2 - y^2 + z^2 \neq 0 \Rightarrow y^2 \neq x^2 + z^2$
- $D = \{(x,y,z) \mid y \geq 0,\ y \neq \sqrt{x^2+z^2}\}$

Example. Using the squeeze theorem, we showed that $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0$. Use polar coordinates to again show that this limit is 0. What does the squeeze theorem look like in this case?

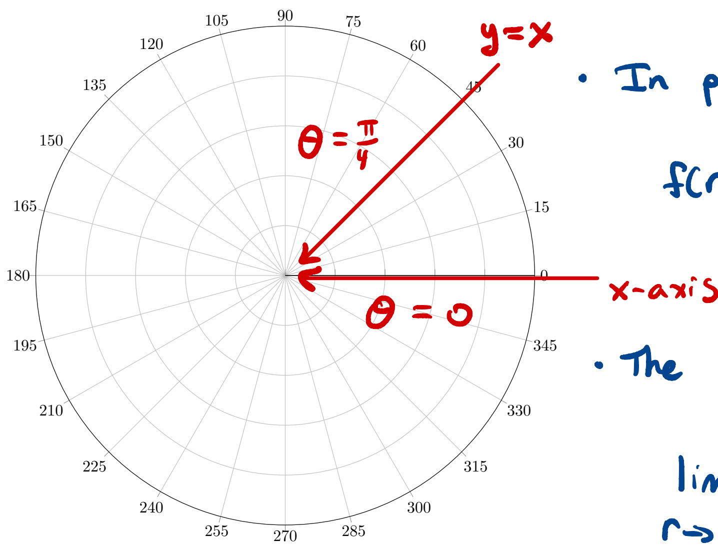
- If we convert (x,y) to (r,θ), then $(x,y)\to(0,0)$ becomes $r\to 0$

- $$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{3\cdot r^2\cos^2\theta\cdot r\sin\theta}{r^2} = \lim_{r\to 0} 3r\cos^2\theta\sin\theta$$

- Now, $-3r \leq 3r\cos^2\theta\sin\theta \leq 3r$

- Since $\lim\limits_{r\to 0} -3r = \lim\limits_{r\to 0} 3r = 0$, $\lim\limits_{r\to 0} 3r\cos^2\theta\sin\theta = 0$ by the squeeze thm.

Example. We showed that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist by showing that $f(x,y)\to 0$ along the x-axis, but $f(x,y)\to \frac{1}{2}$ along the line $y=x$. What does this look like using polar coordinates?



- In polar coordinates,

$$f(r,\theta) = \frac{r\cos\theta\cdot r\sin\theta}{r^2}$$

- The limit becomes

$$\lim_{r\to 0} \cos\theta\sin\theta$$

- Along the line $\theta = 0$, $f(r,0) = \cos(0)\sin(0) = 0$

- Along the line $\theta = \frac{\pi}{4}$, $f(r,\frac{\pi}{4}) = \cos(\frac{\pi}{4})\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{2}{4} = \frac{1}{2}$

Example. Find $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2}$ if it exists.

We can use spherical coordinates:

(in spherical coordinates, $\rho \geq 0$)

$$\lim_{\rho\to 0^+} \frac{\rho \sin\phi\cos\theta \cdot \rho\sin\phi\sin\theta \cdot \rho\cos\phi}{\rho^2}$$

$$\lim_{\rho\to 0^+} \rho \underbrace{\sin^2\phi\cos\phi\cos\theta\sin\theta}_{\text{between } -1 \text{ and } 1}$$

This is 0 by the squeeze theorem since

$$\underset{\substack{\| \\ 0}}{\lim_{\rho\to 0^+} -\rho} \leq \lim_{\rho\to 0^+} \rho \sin^2\phi\cos\phi\cos\theta\sin\theta \leq \underset{\substack{\| \\ 0}}{\lim_{\rho\to 0^+} \rho}$$

Example. Find $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}$ if it exists.

All paths need to be in the domain of the function and approach $(0,0,0)$

Many things to do. Some options:

① Along the x-axis: $f(x,0,0) = \dfrac{0+0+0}{x^2+0^2+0^2} = 0$

② Along the y-axis: $f(0,y,0) = \dfrac{0+0+0}{0+y^2+0} = 0$

③ Along $y=x$ in the xy-plane:

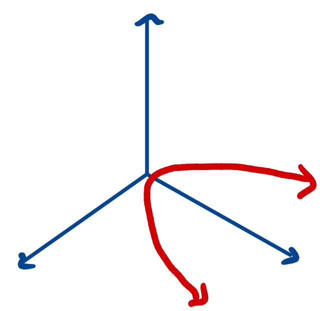
$$f(x,x,0) = \frac{x^2+0+0}{x^2+x^2+0} = \frac{x^2}{2x^2} = \frac{1}{2}$$

④ Along a parametric curve: Doubt you will need this

$x = t^2 \quad y = t^2 \quad z = t$

Important: this curve goes through $(0,0,0)$

$$f(t^2,t^2,t) = \frac{t^4+t^4+t^4}{t^4+t^4+t^4} = 1$$



Conclude: D.N.E.