Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

11.1 Functions of Several Variables

Question. What are some ways to represent a function of two or more variables?

- So far, we have studied functions $f(x,y)$ from their graphs
- Can also study them numerically (using tables), algebraically (using formulas), visually (using level curves/graphs)

Example. The wind-chill index is often used to describe the apparent severity of the cold. This index W is a subjective temperature that depends on the actual temperature T and the wind speed v. So W is a function of T and v, and we can write $W = f(T, v)$. The table below records values of W compiled by the National Weather Service of the US and the Meteorological Service of Canada.

Wind speed (km/h)

Actual temperature (°C)

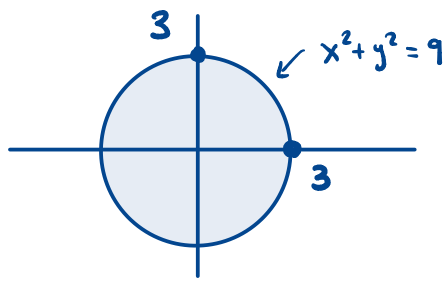
T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

For example, $f(-5, 50) = -15$

Example. Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$

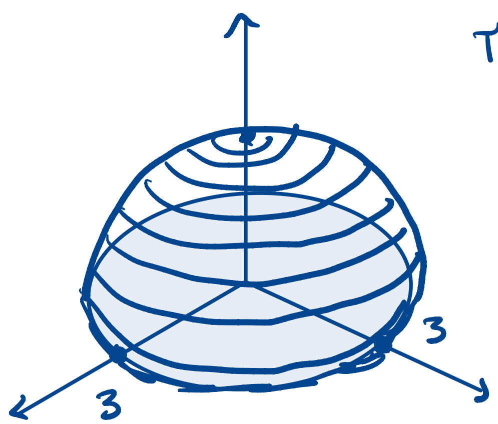
Domain: $D = \{ (x,y) \mid 9 - x^2 - y^2 \geq 0 \}$

$= \{ (x,y) \mid x^2 + y^2 \leq 9 \}$



Range: since $0 \leq x^2+y^2 \leq 9$
$g(x,y)$ varies from
$[0, 3]$

Example. Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.



The graph has equation $z = \sqrt{9-x^2-y^2}$.

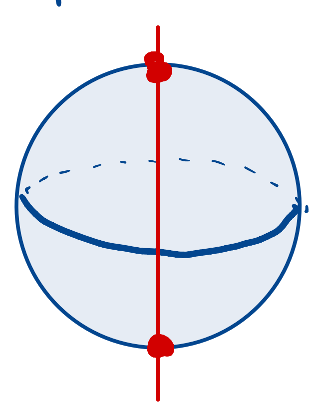
$\Rightarrow z^2 = 9 - x^2 - y^2$

$\Rightarrow x^2 + y^2 + z^2 = 9$ sphere of radius 3

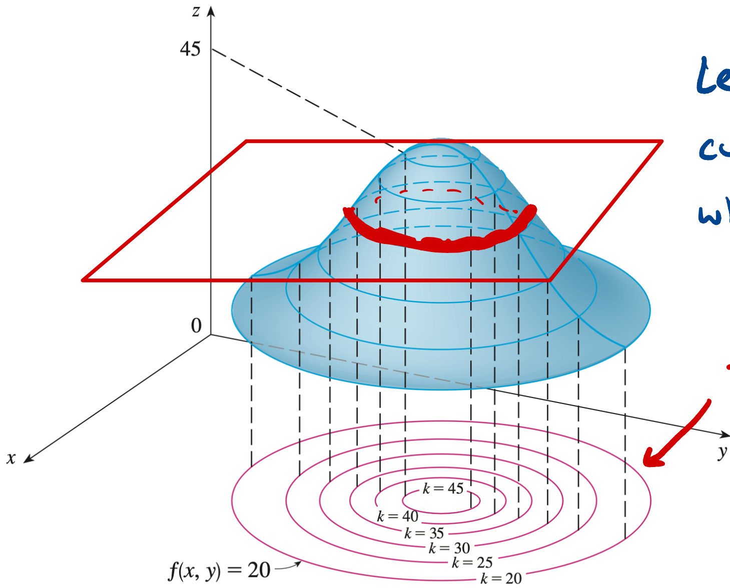
Since $z \geq 0$, we get the top half of the sphere.

Remark. Can an entire sphere be represented by a single function of x and y?

No! A function $f(x,y)$ assigns a pair (x,y) to a unique z-coordinate



Definition. What are the level curves of a function f of two variables?

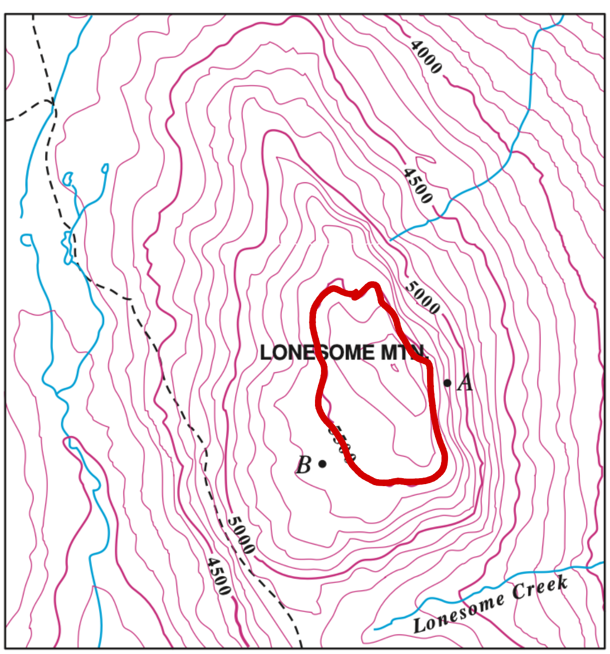


Level curves: these are curves $f(x,y) = k$ where k is some constant

The curve $f(x,y) = 20$ represents all points (x,y) where the height is 20.

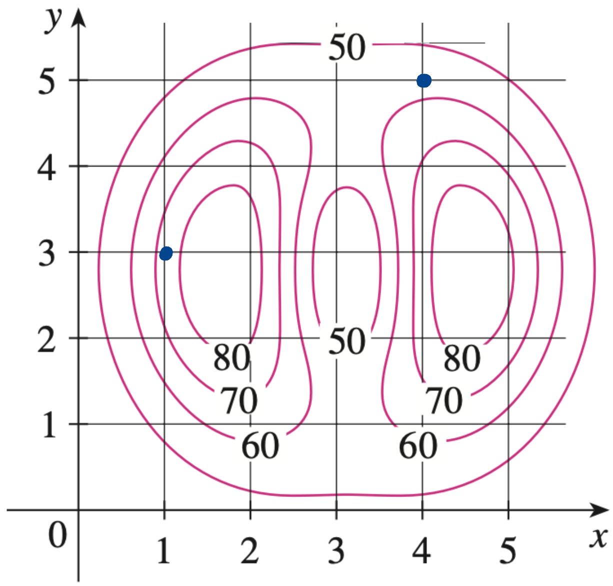
Another way: The level curves are the horizontal cross-sections of the graph projected down to the xy-plane.

Example. What is a common example of level curves?



Walking along a contour line keeps you at constant elevation.

Example. A contour map for a function f is shown below. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.



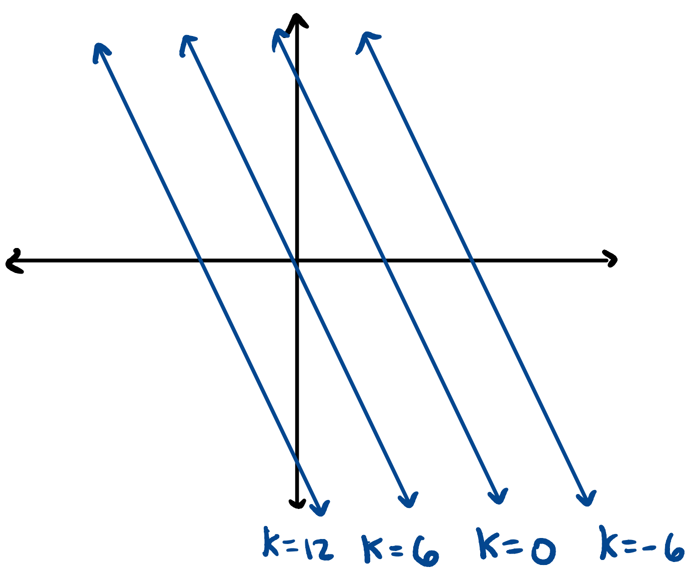
(1, 3) lies between the level curves with z-values 70 and 80

$f(1,3) \approx 73$

(4,5) lies between the level curves with z-values 50 and 60

$f(4,5) \approx 56$

Example. Sketch the level curves of the function $f(x,y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.



The level curves are

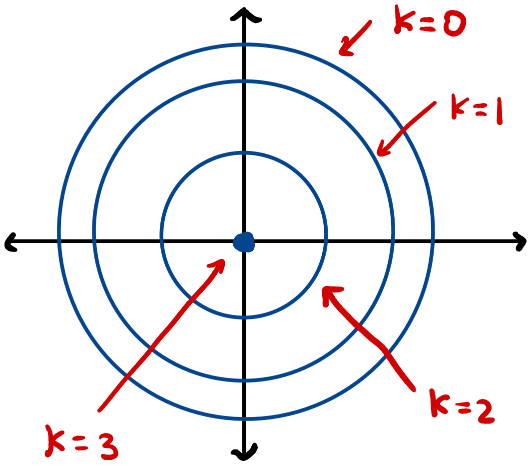
$$6 - 3x - 2y = k$$

$$\Rightarrow \quad 2y = -3x + 6 - k$$

$$\Rightarrow \quad y = -\frac{3}{2}x + \frac{6-k}{2}$$

A family of lines with slope $-3/2$

Example. Sketch the level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3$.



Imagine pulling this up to a hemisphere.

The level curves are

$$\sqrt{9-x^2-y^2} = k$$

$$\iff 9-x^2-y^2 = k^2$$

$$\iff x^2+y^2 = 9-k^2$$

family of circles with center $(0,0)$ and radius $\sqrt{9-k^2}$

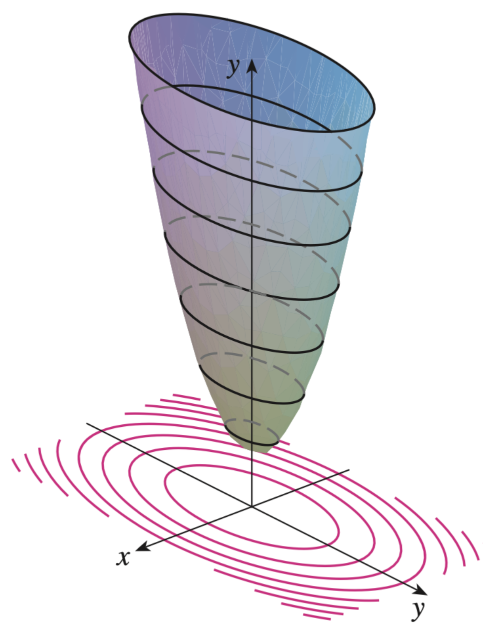
Example. Sketch some level curves of the function $h(x, y) = 4x^2 + y^2 + 1$.

The level curves are

$$4x^2+y^2+1 = k$$

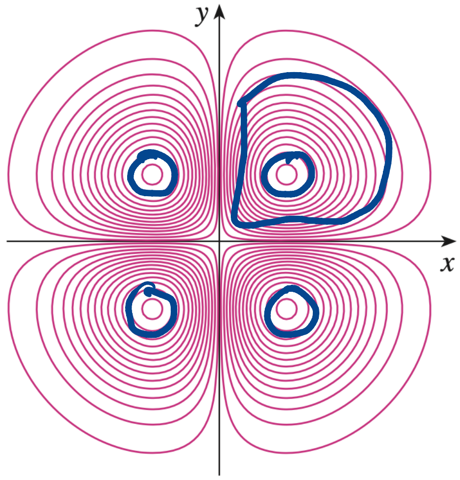
$$\iff \frac{x^2}{\frac{1}{4}(k-1)} + \frac{y^2}{k-1} = 1$$

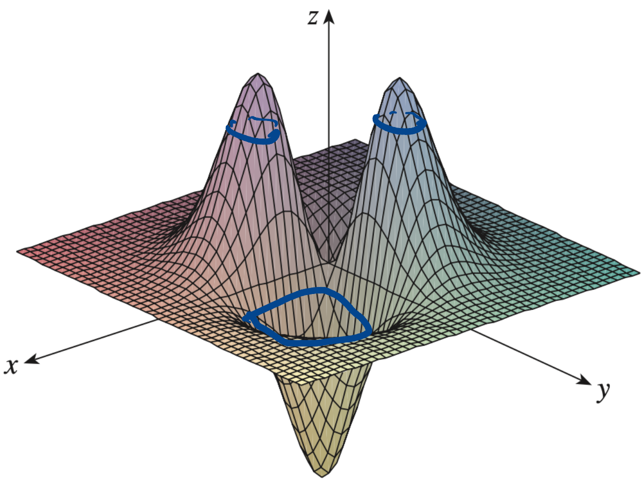
A family of ellipses with semiaxes $\frac{1}{2}\sqrt{k-1}$ and $\sqrt{k-1}$



(Review the equations for types of plane curves: ellipses, hyperbolas, etc...)

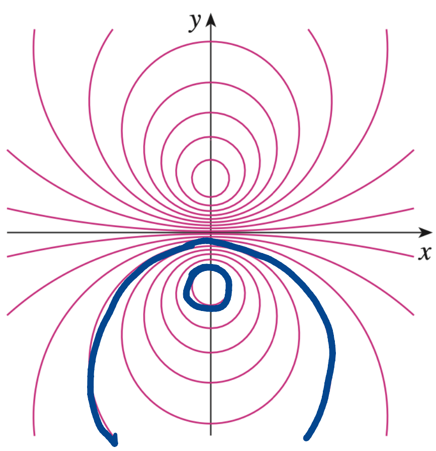
Example. Examine the level curves of the function $f(x, y) = -xye^{-x^2-y^2}$.

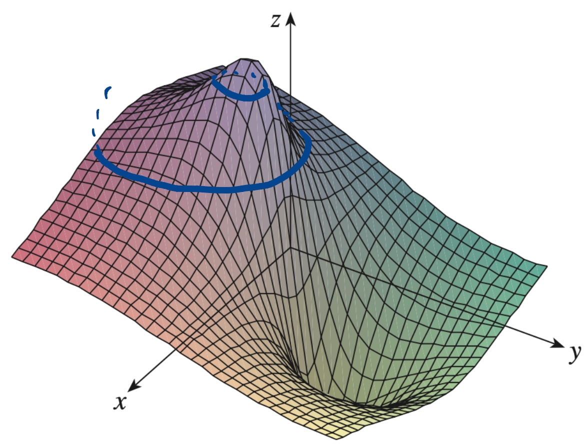




(Question: Match graph of f(x,y) to its level curves)

Example. Examine the level curves of the function $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$.





Definition. What is a function of three variables? Give an example.

A rule $f(x,y,z)$ that assigns a triple (x,y,z) in a domain in $\mathbb{R}^3$ to a unique real number.

e.g. The temperature $T(x, y, t)$ of a point on the earth

longitude latitude time

Example. Find the domain of $f(x, y, z) = \ln(z - y) + xy \sin z$.

We need $z - y > 0$

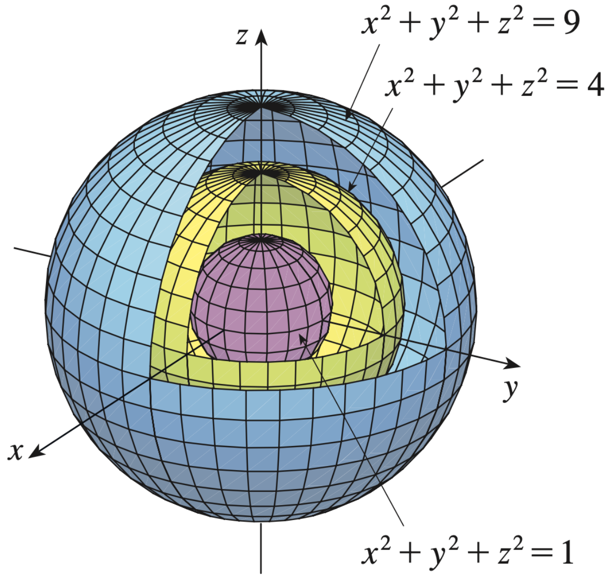
$$D = \{ (x,y,z) \in \mathbb{R}^3 \mid z > y \}$$

Definition. What are the level surfaces of a function f of three variables?

- The graph of $f(x,y,z)$ is a four-dimensional object
- The level surfaces are surfaces in $\mathbb{R}^3$ with

  $f(x,y,z) = k$
- If a point moves along a level surface, $f(x,y,z)$ is constant.

Example. Find the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.



- The level surfaces are $x^2+y^2+z^2=k$ for $k>0$
- These form a family of spheres of radius $\sqrt{k}$
- The value of $f(x,y,z)$ is constant on each sphere.

(The 3D graph of $f(x,y)$ has level curves
The 4D graph of $f(x,y,z)$ has level surfaces)

Definition. What is a function of n variables? Give an example.

Remark. How can we use vector notation to write a function of n variables more compactly?