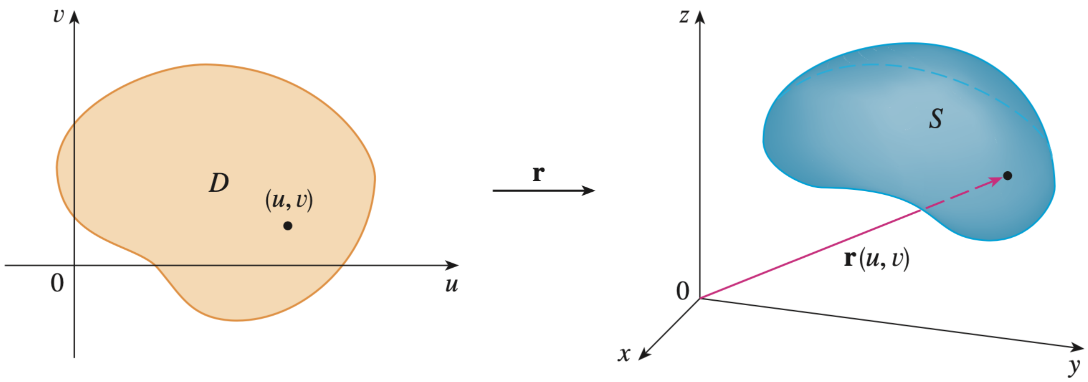
Lecture Notes
Math 2400 - Calculus III
Spring 2024

Name: Champ

10.5 Parametric Surfaces

Definition. What is a parametric surface?

Distinguish this from the graph of a function $f(x,y)$



Space curve: vector function $\vec{r}(t)$ of one parameter

Parametric surface: vector function $\vec{r}(u,v)$ of two parameters

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

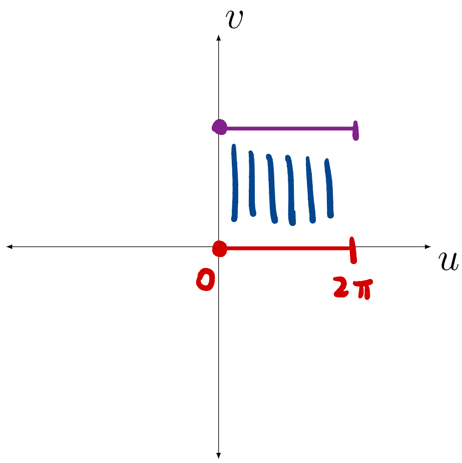
As the parameters u and v vary over a domain D, the points $(x,y,z) \in \mathbb{R}^3$ such that

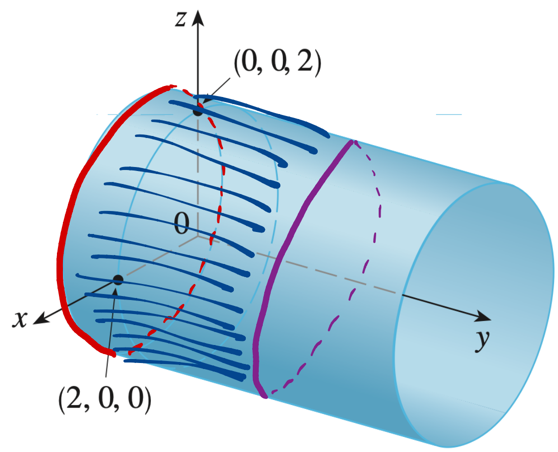
$$x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)$$

trace out a surface.

Example. Identify and sketch the surface with vector equation

$$\vec{r}(u, v) = 2\cos u\vec{i} + v\vec{j} + 2\sin u\vec{k}$$

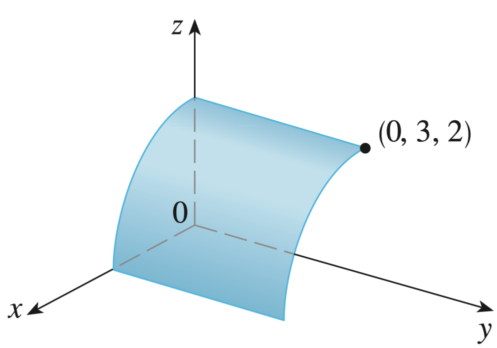




- For a fixed v, as u varies from 0 to 2π, we get a circle of radius 2 parallel to xz-plane.

- As we vary v, we get a whole cylinder

Question. How can we modify the previous example to obtain a quarter-cylinder with length 3?

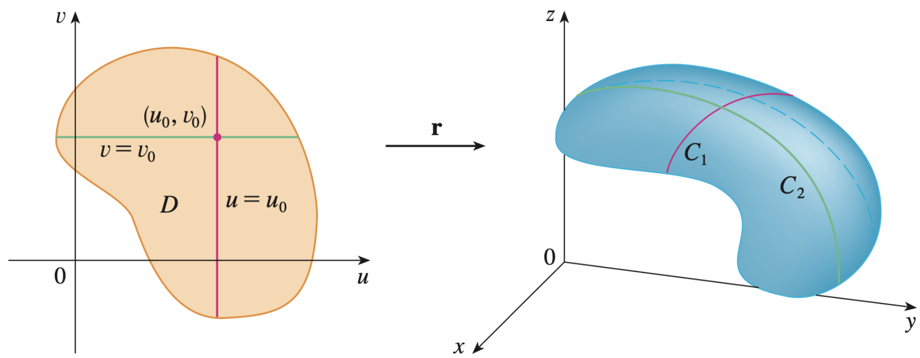


We can restrict u, v in the parameter domain

$$0 \leq u \leq \pi/2$$

$$0 \leq v \leq 3$$

Definition. What are grid curves? → Help analyze parametric surfaces



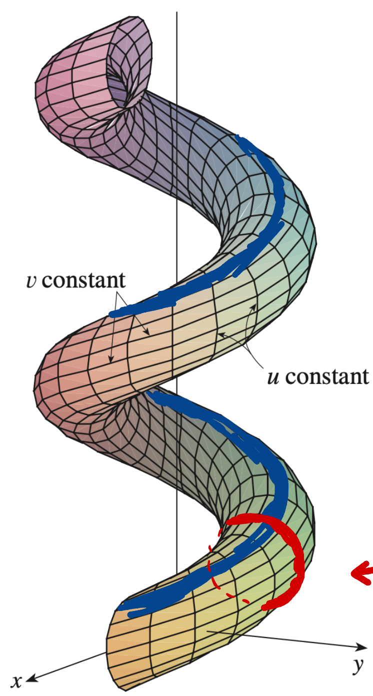
Fix u: The line $u = u_0$ traces a curve C_1 as v varies

Fix v: The line $v = v_0$ traces a curve C_2 as u varies

Example. The surface

$$\vec{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

is graphed below. Which grid curves have u constant? Which have v constant?



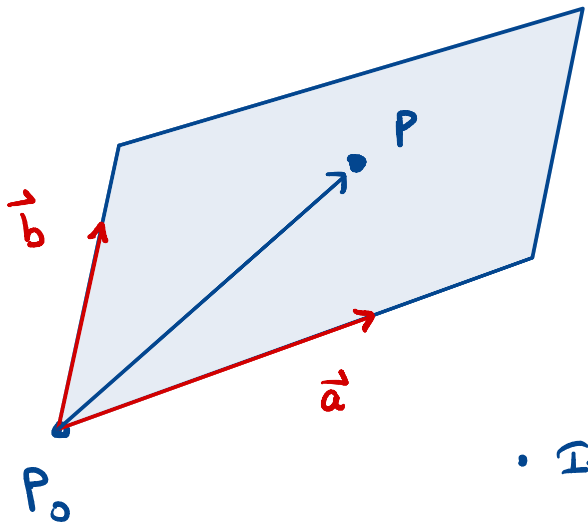
Fix v (e.g. set $v=0$)

- $\vec{r}(u,v) = \langle 2\cos(u), 2\sin(u), u+1 \rangle$
- As u varies, traces out a helix

Fix u (e.g. set $u = \pi/2$)

- $\vec{r}(u,v) = \langle 0, 2+\sin v, \pi/2 + \cos v \rangle$
- traces out a circle of radius 1

Example. Find a vector function that represents the plane that passes through the point P_0 with position vector r_0 and that contains two non-parallel vectors $\vec{a}$ and $\vec{b}$.



- For any point P in the plane, we can find scalars u, v so that

$$\overrightarrow{P_0P} = u\vec{a} + v\vec{b}$$

- If $\vec{r}_0$ is the position vector of P_0 and $\vec{r}$ is the position vector of P, then

$$\vec{r} = \vec{r}_0 + \overrightarrow{P_0P} = \vec{r}_0 + u\vec{a} + v\vec{b}$$

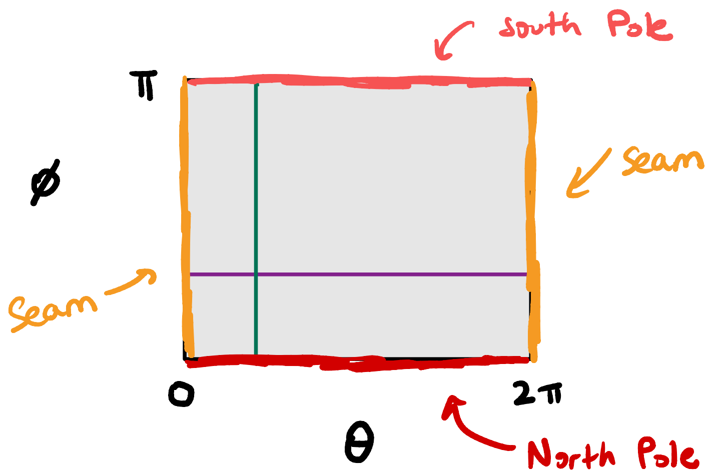
$$\Rightarrow \boxed{\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}}$$

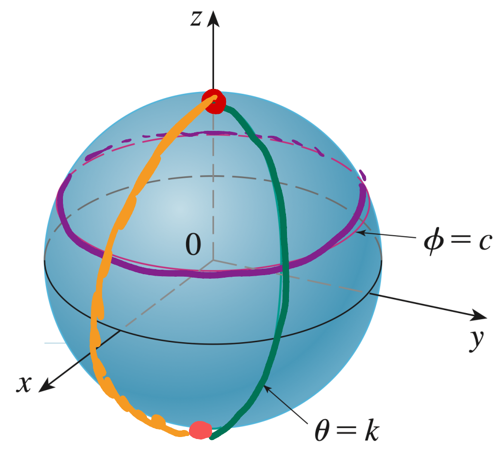
$$\vec{r}(u,v) = \langle x_0, y_0, z_0 \rangle + u\langle a_1, a_2, a_3 \rangle + v\langle b_1, b_2, b_3 \rangle$$

$$\vec{r}(u,v) = \langle x_0 + ua_1 + vb_1,\ y_0 + ua_2 + vb_2,\ z_0 + ua_3 + vb_3 \rangle$$

$\rho = a$

Example. Find a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$. What are the grid curves?





From spherical coordinates,

$$x = a \sin\phi \cos\theta$$
$$y = a \sin\phi \sin\theta$$
$$z = a \cos\phi$$

$$\vec{r}(\theta, \phi) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$$

Q: How to parametrize top half of sphere?

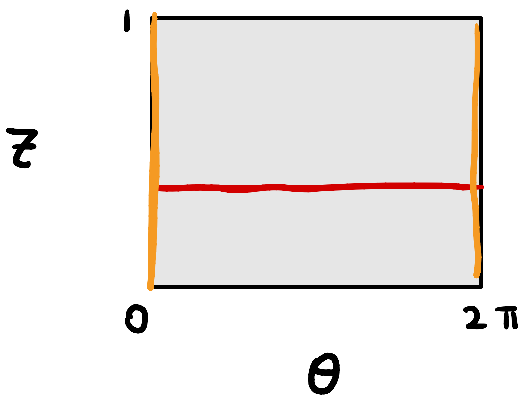
A: $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$

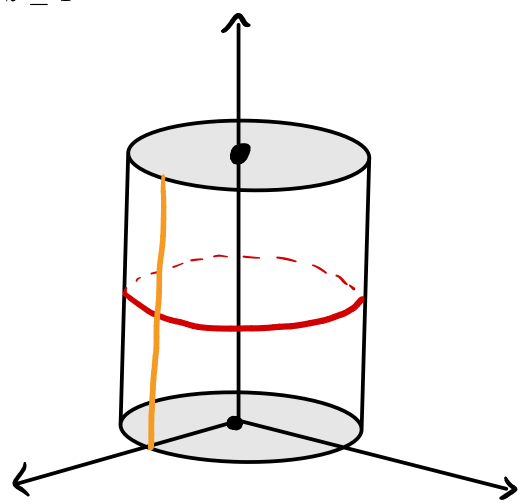
Rmk: In topology, this is called an identification space.

Sphere is obtained from the rectangle by gluing points together.

Example. Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \qquad 0 \le z \le 1$$





- From cylindrical coordinates, we know $r^2 = 4 \Rightarrow r = 2$
- To trace out the circles, $x = 2\cos\theta$, $y = 2\sin\theta$, $z = ?$
- Letting $0 \le z \le 1$ $\vec{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle$.

Remark. Suppose a surface S is given as the graph of a function of x and y, that is, with an equation of the form $z = f(x, y)$. How can we view S as a parametric surface?

If a surface is the graph of a function $f(x,y)$,

we can choose

$$x = x \qquad y = y \qquad z = f(x,y)$$

$$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$$

Example. Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

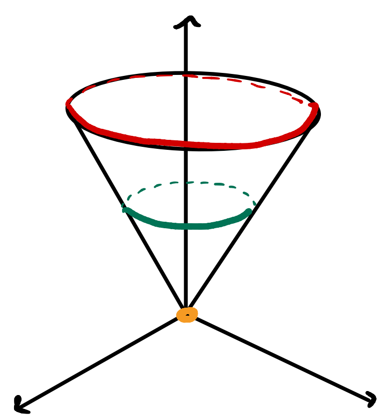
- This is the graph of a function
- From the above,

$$\vec{r}(x,y) = \langle x, y, x^2 + 2y^2 \rangle$$

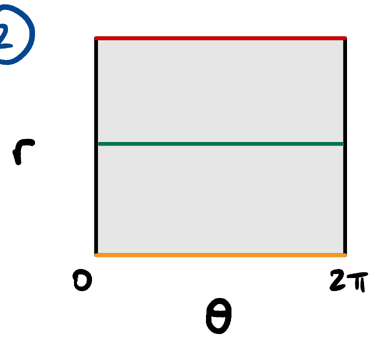
Example. Find a parametric representation for the top half of the cone $z^2 = 4x^2 + 4y^2$.

1. The top half of the cone is $z = \sqrt{4(x^2+y^2)} = 2\sqrt{x^2+y^2}$

$$\vec{r}(x,y) = \langle x, y, 2\sqrt{x^2+y^2} \rangle$$



2.



Using cylindrical coordinates

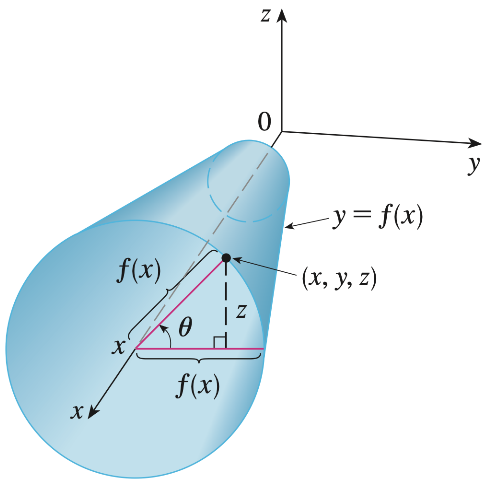
$x = r\cos\theta \qquad y = r\sin\theta \qquad z = ?$

Q: How high should the circle of radius r be?

We know $z^2 = 4r^2 \Rightarrow z = 2r$

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r \rangle$$

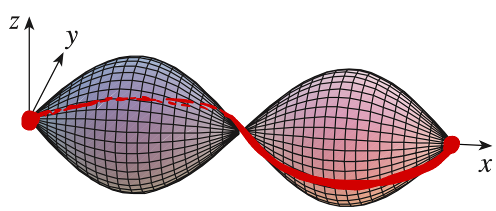
Definition. What is a surface of revolution? How can we represent a surface of revolution parametrically?



- Given a curve $y = f(x)$, we can rotate this curve, say, around the x-axis.
- We can describe the surface of revolution parametrically by

$$\vec{r}(x, \theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$$

Example. Find parametric equations for the surface generated by rotating the curve $y = \sin x, 0 \le x \le 2\pi$ about the x-axis. Use these equations to graph the surface of revolution.

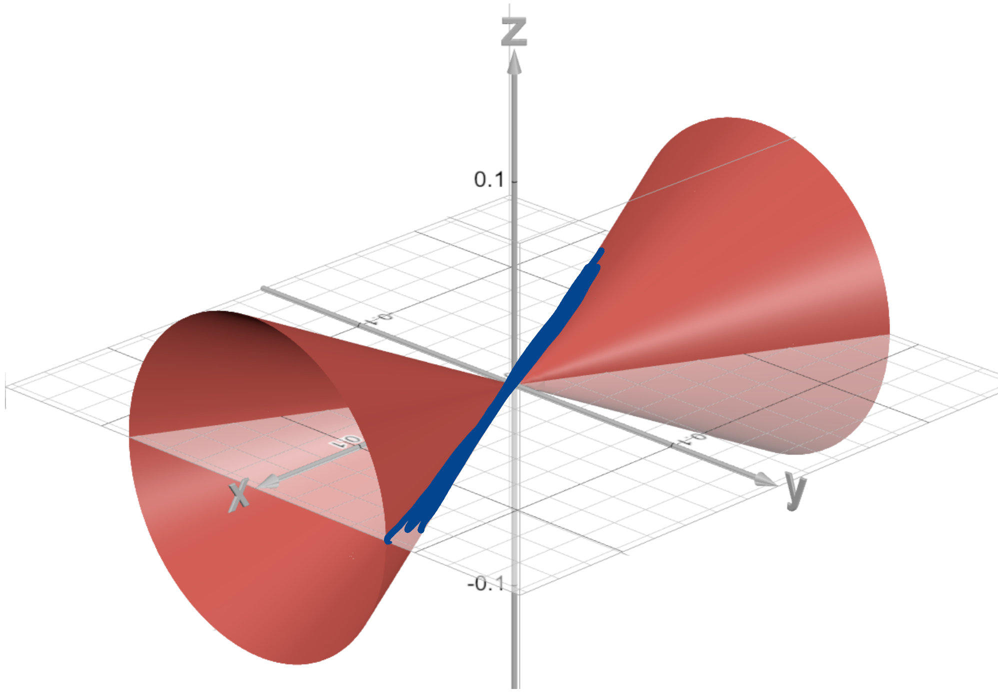


$$\vec{r}(x, \theta) = \langle x, \sin(x)\cos(\theta), \sin(x)\sin(\theta) \rangle$$

$$0 \le x \le 2\pi$$

$$0 \le \theta \le 2\pi$$

Example. Find a parametric representation for the cone $x^2 = 4(y^2 + z^2)$.



- Observation: this is a surface of revolution.
- To find the function that we rotate, set $z=0$.

$$x^2 = 4y^2 \Rightarrow y^2 = \frac{1}{4}x^2 \Rightarrow y = \pm \frac{1}{2}x$$

We can choose the line $y = \frac{1}{2}x$

- $\vec{r}(x,\theta) = \langle x, \frac{1}{2}x\cos(\theta), \frac{1}{2}x\sin(\theta) \rangle$