Lecture Notes
Math 2400 - Calculus III
Spring 2024

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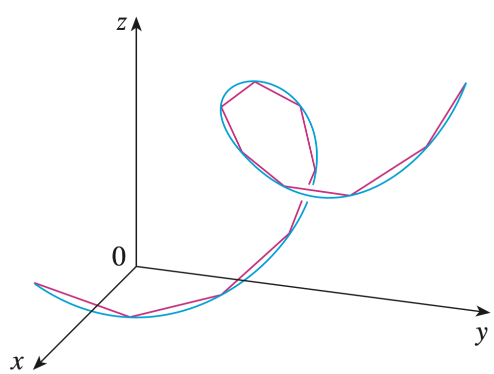
10.3 Arc Length

Calc II

Question. How do we find the length of a plane curve with parametric equations $x = f(t), y = g(t)$ for $a \le t \le b$?

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Definition. How do we find the length of a space curve?



If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

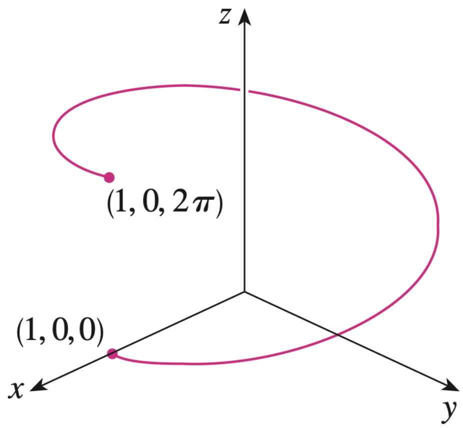
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Remark. How can we write the arc length formula more compactly?

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Example. Find the length of the arc of the circular helix with vector equation $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.



- $L = \int_a^b |r'(t)|\, dt$
- $r'(t) = \langle -\sin t, \cos t, 1 \rangle$
- $|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}$
  $= \sqrt{1+1} = \sqrt{2}$

- Find a and b: At $(1,0,0)$, $t = 0$
  At $(1,0,2\pi)$, $t = 2\pi$

- $L = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\,(2\pi - 0) = 2\sqrt{2} \cdot \pi$

Definition. Can a single curve C be represented by more than one vector function? What is a parametrization? Does arc length depend on the parametrization of C?

1. Yes. $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle \quad 1 \le t \le 2$
   $\vec{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \quad 0 \le u \le \ln 2$
   $\Big\}$ same curve

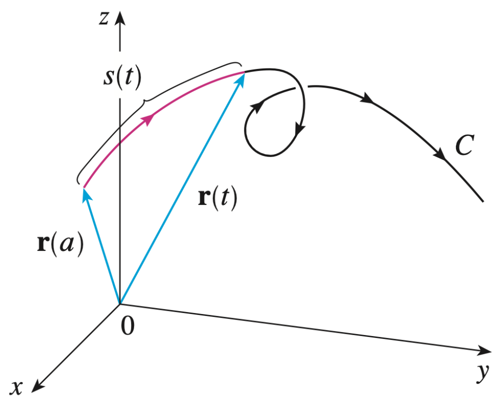
2. A way to represent a curve ↗↗

3. Arc length doesn't depend on parametrization

Definition. Suppose that C is a curve given by a vector function

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \qquad a \le t \le b$$

where $\vec{r}\,'$ is continuous and C is traversed exactly once as t increases from a to b. What is the arc length function $s(t)$?



The function $s(t)$ measures the arc length from starting point a to the input t.

$$s(t) = \int_a^t |r'(u)|\, du$$

Remark. Why is it often useful to parametrize a curve with respect to arc length?

- Goal: remove "time" from the parametrization and instead use something intrinsic to the curve.

e.g. time.

- Idea: let $\vec{r}(t)$ be parametrized with variable t.
  If $s(t)$ is the arc length function, solve for t in terms of s ... $\boxed{t(s)}$
- Then $\vec{r}(t(3))$ is the position 3 units along the curve.

Example. Reparametrize the helix $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

- The point $(1,0,0)$ corresponds to $t=0$
- We have shown $|\vec{r}\,'(t)| = \sqrt{2}$
- $s = \int_0^t |r'(u)|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\, t$
- Hence $t = s/\sqrt{2}$
- Thus $\vec{r}(t(s)) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$