Lecture Notes
Math 2400 - Calculus III
Spring 2024

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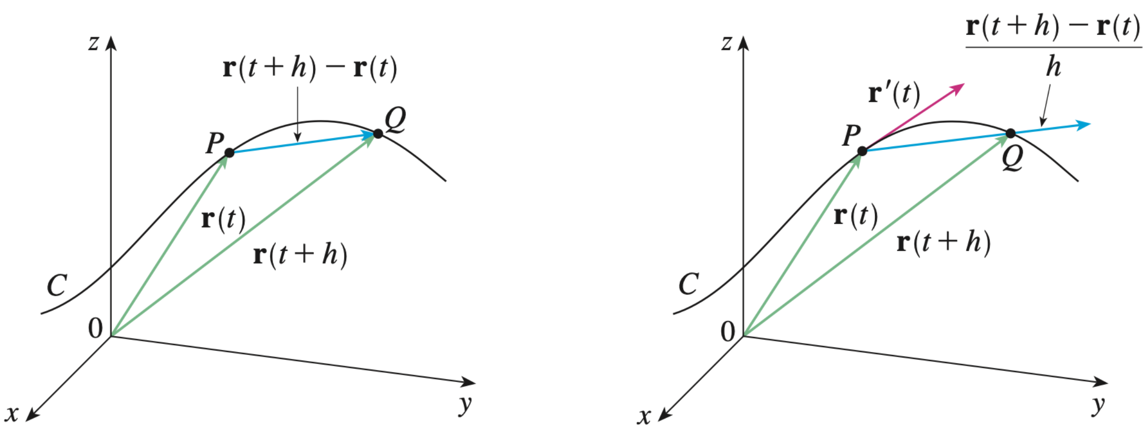
10.2 Derivatives and Integrals of Vector Functions

Definition. What is the derivative of a vector function $\vec{r}(t)$?

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Question. Explain the derivative of a vector function geometrically.



(a) The secant vector (b) The tangent vector

- $\overrightarrow{PQ}$ represents $\vec{r}(t+h) - \vec{r}(t)$ ("secant vector")
- As $h \to 0$:
  - Q approaches P
  - $\vec{r}(t+h) - \vec{r}(t)$ approaches the 0 vector
  - The scalar multiple $\frac{1}{h}(\vec{r}(t+h) - \vec{r}(t))$ approaches a vector $\vec{r}\,'(t)$ that lies on the tangent line.

Definition. What is the unit tangent vector?

Tangent vector at P: $\vec{r}\,'(t)$

Tangent line at P: Line through P parallel to $\vec{r}\,'(t)$

Unit tangent vector: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Theorem. If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, g, and h are differentiable functions, show that $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

$\boxed{\text{To find } \vec{r}\,'(t)\text{, differentiate each component}}$

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\vec{r}(t+\Delta t) - \vec{r}(t)\right]$$

Definition of derivative

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\langle f(t+\Delta t) - f(t),\ g(t+\Delta t) - g(t),\ h(t+\Delta t) - h(t)\rangle\right]$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t},\ \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t},\ \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \right\rangle$$

$$= \langle f'(t),\ g'(t),\ h'(t) \rangle$$

Example.

(a) Find the derivative of $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \sin 2t\vec{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

(a) $\vec{r}\,'(t) = \langle 3t^2, -te^{-t} + e^{-t}, 2\cos 2t \rangle$

(b) $\vec{T}(0) = \dfrac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \dfrac{\langle 0, 1, 2\rangle}{\sqrt{0^2+1^2+2^2}} = \dfrac{\langle 0, 1, 2\rangle}{\sqrt{5}}$

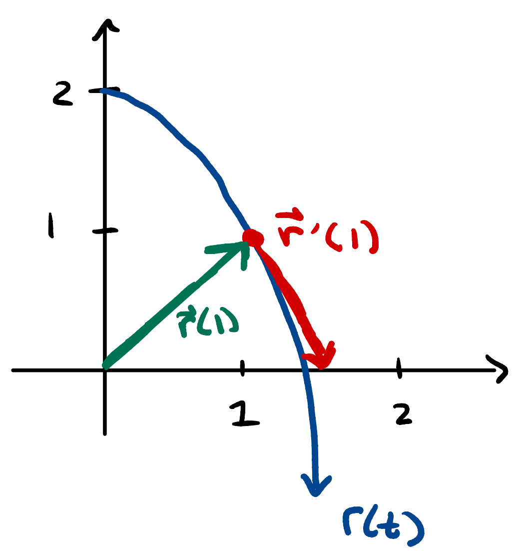
$= \langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

Example. For the curve $\vec{r}(t) = \sqrt{t}\vec{i} + (2-t)\vec{j}$, find $\vec{r}\,'(t)$ and sketch the position vector $\vec{r}(1)$ and the tangent vector $\vec{r}\,'(1)$.

$\vec{r}\,'(t) = \langle \frac{1}{2}t^{-1/2}, -1 \rangle$

$\vec{r}\,'(1) = \langle \frac{1}{2}, -1 \rangle$

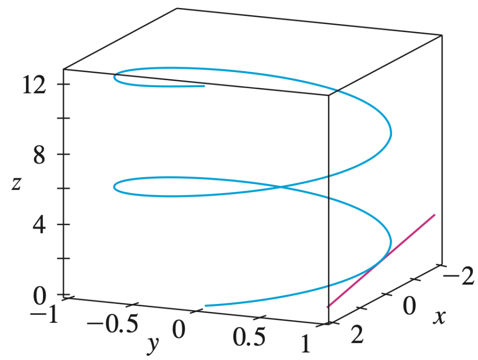
$\vec{r}(1) = \langle 1, 1 \rangle$



Example. Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.



- $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle$
- $\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1 \rangle$
- Need to find t corresponding to the point $(0, 1, \pi/2)$

  $\Rightarrow t = \pi/2$

- $\vec{r}\,'(\pi/2) = \langle -2, 0, 1 \rangle$
- $\vec{L}(t) = \vec{L_0} + t\vec{v}$
- $\vec{L}(t) = \langle 0, 1, \pi/2 \rangle + t\langle -2, 0, 1 \rangle$

$$\boxed{\begin{aligned} x &= -2t \\ y &= 1 \\ z &= \frac{\pi}{2} + t \end{aligned}}$$

Definition. What is the second derivative of a vector function $\vec{r}$? What is the second derivative of the function in the previous example?

- $\vec{r}\,''(t) = (\vec{r}\,')'(t)$
- In the above, $\vec{r}\,''(t) = \langle -2\cos t, -\sin t, 0 \rangle$

Theorem. Suppose $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Find differentation formulas for the following expressions.

1. $\dfrac{d}{dt}[\vec{u}(t)+\vec{v}(t)] =$ $\vec{u}\,'(t)+\vec{v}\,'(t)$

2. $\dfrac{d}{dt}[c\vec{u}(t)] =$ $c\,\vec{u}\,'(t)$

3. $\dfrac{d}{dt}[f(t)\vec{u}(t)] =$ $f'(t)\,\vec{u}(t) + f(t)\,\vec{u}\,'(t)$

4. $\dfrac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] =$ $\vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

5. $\dfrac{d}{dt}[\vec{u}(t)\times\vec{v}(t)] =$ $\vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$

6. $\dfrac{d}{dt}[\vec{u}(f(t))] =$ $f'(t)\,\vec{u}\,'(f(t))$

"Chain rule"

Example. Show that if $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal for all t. What does this result mean geometrically?

Definition. What is the definite integral of a continuous vector function $\vec{r}(t)$?

Question. How can we extend the Fundamental Theorem of Calculus to continuous vector functions?

Example. Find $\int_0^{\pi/2} \vec{r}(t)\, dt$ if $\vec{r}(t) = 2\cos t\vec{i} + \sin t\vec{j} + 2t\vec{k}$.