Lecture Notes
Math 2400 - Calculus III
Spring 2024

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10.1 Vector Functions and Space Curves

Definition. What is a vector-valued function? What are the component functions of a vector-valued function?

A (3-D) vector-valued function is a function $\vec{r}(t): \mathbb{R} \to V_3$ of the form

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where $f(t)$, $g(t)$, $h(t)$ are real-valued functions called the component functions of $\vec{r}(t)$.

Example. Let $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$. What are the component functions? What is the domain of $\vec{r}(t)$?

Component functions: $f(t) = t^3$ $\quad g(t) = \ln(3-t)$ $\quad h(t) = \sqrt{t}$

$t \in (-\infty, \infty)$ $\quad 3 - t > 0$ $\quad t \geq 0$

Domain: $[0, 3)$

Definition. How to take a limit of a vector-valued function?

- Take the limits of the component functions
- $\lim\limits_{t \to a} \vec{r}(t) = \left\langle \lim\limits_{t \to a} f(t), \lim\limits_{t \to a} g(t), \lim\limits_{t \to a} h(t) \right\rangle$

provided these limits exist.

Example. Find $\lim_{t\to 0} \vec{r}(t)$, where $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \dfrac{\sin t}{t}\vec{k}$.

$$\lim_{t\to 0} \vec{r}(t) = \left[\lim_{t\to 0} 1+t^3\right]\vec{i} + \left[\lim_{t\to 0} te^{-t}\right]\vec{j} + \left[\lim_{t\to 0} \frac{\sin t}{t}\right]\vec{k}$$

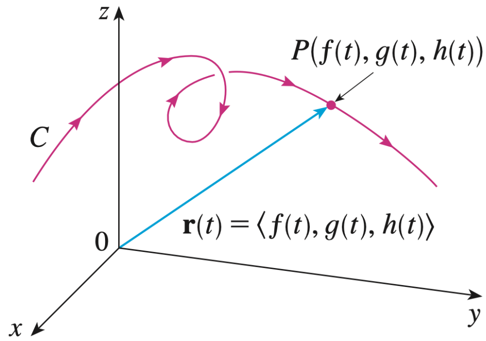
$$= \vec{i} + \vec{k}$$

$$= \langle 1, 0, 1 \rangle$$

Definition. What does it mean for a vector function $\vec{r}(t)$ to be continuous at a?

- $\vec{r}(t)$ is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$
- $\vec{r}(t)$ is continuous at a $\iff$ its component functions are continuous at a.

Definition. What is a space curve?



As t varies, the tip of a continuous vector function $\vec{r}(t)$ traces out a space curve C.

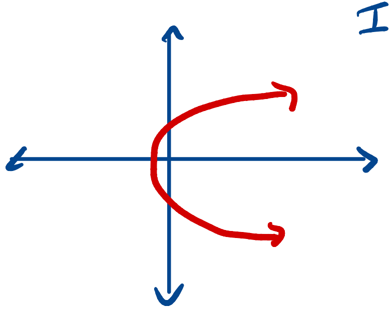
Parametric Equations of C: $x = f(t)$ $y = g(t)$ $z = h(t)$

Example. Describe the curve defined by the vector function $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t \rangle$.

- $\vec{r}(t) = \langle 1, 2, -1 \rangle + t \langle 1, 5, 6 \rangle$
  $= \vec{r}_0 + t\vec{v}$

- This is a line passing through $(1, 2, -1)$ parallel to the vector $\langle 1, 5, 6 \rangle$

Remark. Plane curves can also be represented in vector notation. How would we write the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ in vector notation?

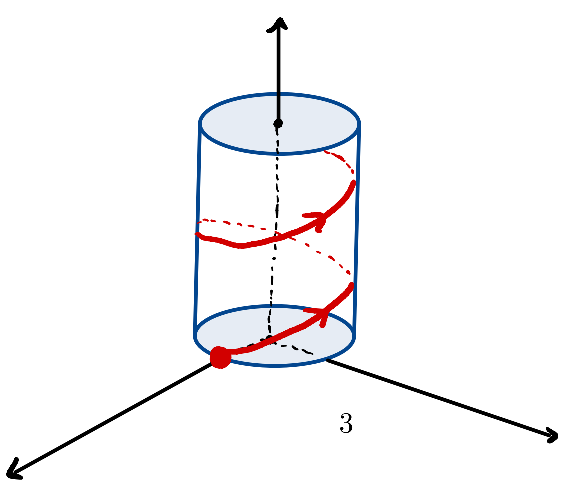


In 2-D space, we only have two component functions

$\vec{r}(t) = \langle t^2 - 2t, t+1 \rangle$

Example. Sketch the curve whose vector equation is $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$.

- The parametric equations are $x = \cos t$ $y = \sin t$ $z = t$
- Note: $x^2 + y^2 = 1 \Rightarrow$ The curve lies on a cylinder
- We traverse the cylinder counterclockwise and slowly increase z



Example. Find a vector equation for the line segment that joins the point $P(1, 3, -2)$ to the point $Q(2, -1, 3)$.

- To join the tip of vector $\vec{r}_0$ to the tip of vector $\vec{r}_1$ we can use $\boxed{\vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1, \quad 0 \le t \le 1}$

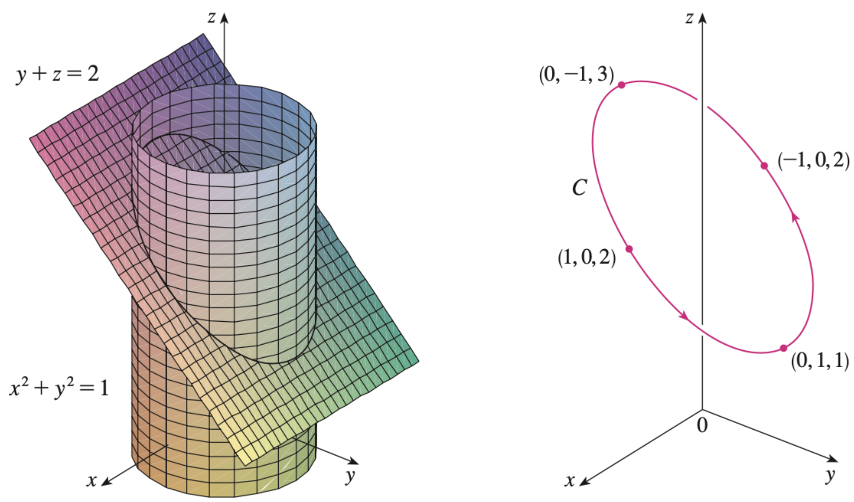
  At $t=0$, at $\vec{r}_0$

  At $t=1$, at $\vec{r}_1$

- In this case, $\vec{r}(t) = (1-t)\langle 1, 3, -2\rangle + t\langle 2, -1, 3\rangle, \quad 0 \le t \le 1$

$$= \langle 1+t,\ 3-4t,\ -2+5t\rangle, \quad 0 \le t \le 1$$

Example. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.



- The plane and the cylinder intersect in an ellipse C
- If we project C to the xy-plane, we get a circle

$$\Rightarrow \quad x = \cos t \qquad y = \sin t \qquad z = ? \qquad 0 \le t \le 2\pi$$

- Since C is on the plane, $z = 2 - y = 2 - \sin t$

- $\vec{r}(t) = \langle \cos t,\ \sin t,\ 2 - \sin t\rangle \qquad 0 \le t \le 2\pi$