

Taylor Polynomials and Taylor's Inequality

Taylor's Inequality: Suppose there is a constant M such that $|f^{(n+1)}(x)| \leq M$ on the interval $[a - d, a + d]$. Then, for every x in $[a - d, a + d]$,

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \leq \frac{M}{(n+1)!} d^{n+1}.$$

Alternating Series Estimation Theorem: For an alternating series whose term magnitudes decrease to 0, the error is at most the first omitted term.

Problem 1.

Consider

$$f(x) = \ln x, \quad a = 1, \quad n = 3, \quad 0.8 \leq x \leq 1.2.$$

- Find the third-degree Taylor polynomial $T_3(x)$ centered at $a = 1$.
- Use Taylor's Inequality to find a bound for $|R_3(x)|$ on the interval $0.8 \leq x \leq 1.2$.
- Use your polynomial to approximate $\ln(1.2)$, and state a guaranteed upper bound on the error.

Problem 2.

Use Taylor's Inequality to determine the minimum degree n such that the Maclaurin polynomial $T_n(x)$ for e^x approximates $e^{0.2}$ to within 0.000001.

Problem 3.

Use the Maclaurin polynomial

$$T(x) = 1 - \frac{x^2}{2!}$$

to approximate $\cos(0.3)$.

- (a) Use the Alternating Series Estimation Theorem to estimate the error in this approximation.
- (b) Use Taylor's Inequality with $n = 2$ to estimate the error in this approximation.
- (c) Use Taylor's Inequality with $n = 3$ to estimate the error in this approximation. (Key idea: for $\cos x$, this polynomial is both $T_2(x)$ and $T_3(x)$.)
- (d) Which remainder estimate is sharpest in this situation? Briefly explain.

Problem 4.

Let $f(x) = e^{3x}$.

- (a) Find the Maclaurin series for $f(x)$ and determine its interval of convergence.
- (b) If $R_n(x) = f(x) - T_n(x)$, show that $\lim_{n \rightarrow \infty} R_n(x) = 0$ for each fixed real number x .
- (c) Explain why this proves that the Maclaurin series for e^{3x} converges to e^{3x} for all real numbers x .