Taylor Polynomials and Taylor's Inequality (Solutions)

Problem 1.

Consider

$$f(x) = \ln x, \qquad a = 1, \qquad n = 3, \qquad 0.8 \le x \le 1.2.$$

(a) Find the third-degree Taylor polynomial $T_3(x)$ centered at $a = 1$.

(b) Use Taylor's Inequality to find a bound for $|R_3(x)|$ on the interval $0.8 \le x \le 1.2$.

(c) Use your polynomial to approximate $\ln(1.2)$, and state a guaranteed upper bound on the error.

Solution.

(a) The formula for the third-degree Taylor polynomial centered at $a = 1$ is

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

Compute derivatives at $a = 1$:

$$\begin{array}{cccc} f(x) & f'(x) = \dfrac{1}{x} & f''(x) = -\dfrac{1}{x^2} & f'''(x) = \dfrac{2}{x^3} \\[2ex] f(1) = 0 & f'(1) = 1 & f''(1) = -1 & f'''(1) = 2 \end{array}$$

Therefore,

$$T_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

(b) We have

$$f^{(4)}(x) = -\frac{6}{x^4}, \qquad |f^{(4)}(x)| = \frac{6}{x^4}.$$

On $[0.8, 1.2]$, this is largest at $x = 0.8$, so

$$M = \frac{6}{(0.8)^4}.$$

Also, $|x-1| \le 0.2$. Hence

$$|R_3(x)| \le \frac{M}{4!}|x-1|^4 \le \frac{6/(0.8)^4}{24}(0.2)^4 \approx 0.000977$$

(c) Using $x = 1.2$,

$$\ln(1.2) \approx T_3(1.2) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} = 0.182667.$$

By part (b), since 1.2 is in $[0.8, 1.2]$ we can guarantee that the error is less than 0.000977.

Problem 2.

Use Taylor's Inequality to determine the minimum degree n such that the Maclaurin polynomial $T_n(x)$ for e^x approximates $e^{0.2}$ to within 0.000001.

Solution.

For $f(x) = e^x$, every derivative is e^x. On $[-0.2, 0.2]$,

$$|f^{(n+1)}(x)| \leq e^{0.2}.$$

So Taylor's Inequality gives

$$|R_n(0.2)| \leq \frac{e^{0.2}}{(n+1)!}(0.2)^{n+1}.$$

We want

$$\frac{e^{0.2}(0.2)^{n+1}}{(n+1)!} < 0.000001$$

Check nearby values:

$$n = 4: \qquad \frac{e^{0.2}(0.2)^5}{5!} \approx 0.000003257675 > 0.000001,$$

$$n = 5: \qquad \frac{e^{0.2}(0.2)^6}{6!} \approx 0.000000108589 < 0.000001.$$

Therefore, the minimum degree is

$$\boxed{n = 5}.$$

Problem 3.

Use the Maclaurin polynomial

$$T(x) = 1 - \frac{x^2}{2!}$$

to approximate $\cos(0.3)$.

(a) Use the Alternating Series Estimation Theorem to estimate the error in this approximation.

(b) Use Taylor's Inequality with $n = 2$ to estimate the error in this approximation.

(c) Use Taylor's Inequality with $n = 3$ to estimate the error in this approximation. (Key idea: for $\cos x$, this polynomial is both $T_2(x)$ and $T_3(x)$.)

(d) Which remainder estimate is sharpest in this situation? Briefly explain.

Solution.

Since $T(0.3) = 0.955$, we use 0.955 to approximate $\cos(0.3)$.

(a) The Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

At $x = 0.3$, this is an alternating series with decreasing term magnitudes, so by the Alternating Series Estimation Theorem,

$$|R(0.3)| \leq \frac{(0.3)^4}{4!} = \frac{0.0081}{24} = 0.0003375.$$

(b) If we view $T(x)$ as $T_2(x)$, then $f(x) = \cos x$ has

$$f^{(3)}(x) = \sin x.$$

On the interval $[-0.3, 0.3]$, we may take $M = 1$. Taylor's Inequality gives

$$|R_2(0.3)| \leq \frac{M}{3!}(0.3)^3 = \frac{1}{6}(0.3)^3 = \frac{0.027}{6} = 0.0045.$$

(c) Since $T(x)$ is also $T_3(x)$ for $\cos x$, we use

$$f^{(4)}(x) = \cos x.$$

Again, on $[-0.3, 0.3]$, we may take $M = 1$. Taylor's Inequality gives

$$|R_3(0.3)| \leq \frac{M}{4!}(0.3)^4 = \frac{1}{24}(0.3)^4 = 0.0003375.$$

(d) The sharpest estimates here are the Alternating Series estimate and the Taylor estimate with $n = 3$, since both give

$$0.0003375.$$

The Taylor estimate with $n = 2$ is much larger.

Problem 4.

Let $f(x) = e^{3x}$.

(a) Find the Maclaurin series for $f(x)$ and determine its interval of convergence.

(b) If $R_n(x) = f(x) - T_n(x)$, show that $\lim_{n\to\infty} R_n(x) = 0$ for each fixed real number x.

(c) Explain why this proves that the Maclaurin series for e^{3x} converges to e^{3x} for all real numbers x.

Solution.

(a) Since $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$, substituting $u = 3x$ gives

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}.$$

To determine the interval of convergence, we can apply the Ratio Test with $a_n = \frac{3^n x^n}{n!}$.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3|x|}{n+1} = 0$$

Since $L < 1$ for every real x, the series converges for all real x. Therefore, the interval of convergence is $(-\infty, \infty)$.

(b) Fix any real number x, and let $d = |x|$. For every t in $[-d, d]$,

$$f^{(n+1)}(t) = 3^{n+1} e^{3t} \quad \Rightarrow \quad |f^{(n+1)}(t)| \le 3^{n+1} e^{3d}.$$

Thus we may take $M = 3^{n+1} e^{3d}$ on the interval $[-d, d]$. By Taylor's Inequality,

$$|R_n(x)| \le \frac{3^{n+1} e^{3d}}{(n+1)!} d^{n+1} = e^{3d} \frac{(3d)^{n+1}}{(n+1)!}.$$

Therefore,

$$0 \le \lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} e^{3d} \frac{(3d)^{n+1}}{(n+1)!} = 0.$$

Since x was arbitrary, we conclude that $R_n(x) \to 0$ for every real number x.

(c) For each n,

$$f(x) = T_n(x) + R_n(x).$$

Taking limits and using part (b), we get

$$f(x) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} T_n(x).$$

Hence the Maclaurin series for e^{3x} converges to e^{3x} for all real x.