

Given a positive integer n , n factorial (written $n!$) is a shorthand for the product of all positive integers less than or equal to the given integer n .

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. (By convention, we say that $0! = 1$.)

1. Evaluate the following.

(a) $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

(b) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

(c) $\frac{6!}{5!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 6$

(d) $\frac{5!}{6!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{6}$

(e) $\frac{102!}{100!} = \frac{102 \cdot 101 \cdot 100!}{100!} = 102 \cdot 101$

2. Simplify each of the following. Assume n is a positive integer.

(a) $\frac{(n+2)!}{n!} = (n+2)(n+1)$

(b) $\frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$

(c) $\frac{(2n+2)!}{(2n)!} = (2n+2)(2n+1)$

(d) $\frac{(2n+2)!}{2n!} = \frac{(2n+2)(2n+1)(2n) \dots (n+1)}{2}$

(e) $\frac{(n!)^2}{((n+1)!)^2} = \frac{n! \cdot n!}{(n+1)! \cdot (n+1)!} = \frac{1}{n+1} \cdot \frac{1}{n+1} = \frac{1}{(n+1)^2}$

2. Series Classification

Determine whether each series converges absolutely, converges conditionally, or diverges. Justify your answer.

1.
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$$

Intro: We will first test for *absolute convergence*. Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n}{n^2 + 1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}.$$

Let $a_n = \frac{n}{n^2+1}$ and $b_n = \frac{1}{n}$. Both are positive, so the Limit Comparison Test applies.

Apply L.C.T.:

$$\lim_{n \rightarrow \infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

Conclusion: Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also diverges by the Limit Comparison Test. The series $\sum a_n$ does *not* converge absolutely.

Intro: The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$ is alternating, so the Alternating Series Test applies.

Apply the A.S.T.: First,

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1/n}{1 + 1/n^2} = 0.$$

Next, b_n is eventually decreasing. To see this, consider

$$f(x) = \frac{x}{x^2 + 1}.$$

Then

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} < 0 \quad \text{for } x > 1.$$

Therefore b_n is decreasing for $n \geq 2$.

Conclusion: The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}$ converges by the Alternating Series Test. Because it converges but does not converge absolutely, it is **conditionally convergent**.

$$2. \sum_{n=1}^{\infty} \frac{1 + 2 \sin n}{n^3}$$

Intro: We will test for *absolute convergence*. Consider the series of absolute values:

$$\sum_{n=1}^{\infty} \left| \frac{1 + 2 \sin n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{|1 + 2 \sin n|}{n^3}.$$

Let $a_n = \frac{|1 + 2 \sin n|}{n^3}$ and $b_n = \frac{3}{n^3}$. Both are positive, so the D.C.T. applies.

Apply the D.C.T.: Since $-1 \leq \sin n \leq 1$, we have

$$|1 + 2 \sin n| \leq 3.$$

Therefore,

$$0 \leq \frac{|1 + 2 \sin n|}{n^3} \leq \frac{3}{n^3}.$$

Also, $\sum_{n=1}^{\infty} \frac{3}{n^3}$ is a convergent p -series with $p = 3 > 1$.

Conclusion: $\sum_{n=1}^{\infty} \frac{|1 + 2 \sin n|}{n^3}$ converges by the Direct Comparison Test. Since the series of absolute values converges, the original series **converges absolutely**.

$$3. \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$$

Intro: We will first test for *absolute convergence*. Consider the series of absolute values:

$$\sum_{n=2}^{\infty} \left| (-1)^{n+1} \frac{\ln n}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n}.$$

Let $f(x) = \frac{\ln x}{x}$. For $x \geq 2$, $f(x) > 0$. Also,

$$f'(x) = \frac{1 - \ln x}{x^2}.$$

Since $1 - \ln x < 0$ when $x > e$, we have $f'(x) < 0$ for $x \geq 3$. Thus f is decreasing on $[3, \infty)$, so the Integral Test applies.

Apply the Integral Test: Now,

$$\int_2^{\infty} \frac{\ln x}{x} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{\ln x}{x} dx = \lim_{b \rightarrow \infty} \frac{(\ln b)^2}{2} - \frac{(\ln 2)^2}{2} = \infty.$$

Conclusion: Therefore, $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges. and the series does *not* converge absolutely.

Intro: The series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$ is alternating, so the Alternating Series Test applies.

Apply the A.S.T.:

1. b_n is eventually decreasing. To see this, consider $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{1 - \ln x}{x^2}$.

Since $f'(x) < 0$ for $x > e$, the sequence b_n is decreasing for all sufficiently large n .

2. $b_n \rightarrow 0$.

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n} \stackrel{\text{L'H}}{=} \lim_{n \rightarrow \infty} \frac{1/n}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Conclusion: Since $b_n \rightarrow 0$ and b_n is eventually decreasing, the series

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$$

converges by the Alternating Series Test. Because it converges but does not converge absolutely, it is **conditionally convergent**.

$$4. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n^2 + 1}}{n + 1}$$

Intro: We will apply the Test for Divergence.

Apply the Test: Let

$$a_n = (-1)^{n+1} \frac{\sqrt{n^2 + 1}}{n + 1}.$$

Then

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 1}}{n + 1} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 \left(1 + \frac{1}{n^2}\right)}}{n \left(1 + \frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{n^2}}}{1 + \frac{1}{n}} = 1.$$

Thus the magnitude of a_n approaches 1, while the factor $(-1)^{n+1}$ causes the terms to alternate in sign. Therefore the terms do not approach 0. In fact, they oscillate between values close to 1 and -1 .

Conclusion: Since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series **diverges** by the Test for Divergence.

3. Ratio Test Practice

1. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{2^n}$

Intro: Test for absolute convergence using the Ratio Test.

Apply the Test: Let

$$a_n = (-1)^n \frac{n}{2^n}.$$

Then

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| (-1)^{n+1} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{(-1)^n n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n+1}{n} \right| \\ &= \lim_{n \rightarrow \infty} \left| (-1) \cdot \frac{1}{2} \cdot \frac{n+1}{n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{n+1}{n} \\ &= \frac{1}{2}. \end{aligned}$$

Conclusion: Since $L = \frac{1}{2} < 1$, the series converges absolutely by the Ratio Test.

$$2. \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

Intro: Test for convergence using the Ratio Test.

Apply the Test: Let

$$a_n = \frac{3^n}{n!}.$$

Then

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right| = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0.$$

Conclusion: Since $L = 0 < 1$, the series converges by the Ratio Test.

$$3. \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

Intro: Test for convergence using the Ratio Test.

Apply the Test: Let

$$a_n = \frac{(n!)^2}{(2n)!}.$$

Then

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{((n+1)n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2(n!)^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(n!)^2} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} \\ &= \frac{1}{4}. \end{aligned}$$

Conclusion: Since $L = \frac{1}{4} < 1$, the series converges by the Ratio Test.

$$4. \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Intro: Test for convergence using the Ratio Test.

Apply the Test: Let

$$a_n = \frac{n!}{n^n}.$$

Then

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right| = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{n(1 + \frac{1}{n})} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}. \end{aligned}$$

So it remains to evaluate

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n.$$

Take the natural logarithm:

$$\ln \left[\left(1 + \frac{1}{n}\right)^n \right] = n \ln \left(1 + \frac{1}{n}\right) = \frac{\ln(1 + \frac{1}{n})}{1/n}.$$

As $n \rightarrow \infty$, this is an indeterminate form of type $0/0$, so by L'Hôpital's Rule,

$$\lim_{n \rightarrow \infty} \frac{\ln(1 + \frac{1}{n})}{1/n} = \lim_{n \rightarrow \infty} \frac{-1/n^2}{-1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} = 1.$$

Thus

$$\ln \left[\left(1 + \frac{1}{n}\right)^n \right] \rightarrow 1.$$

Exponentiating, we obtain

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e.$$

Hence

$$L = \frac{1}{e}.$$

Conclusion: Since $L = \frac{1}{e} < 1$, the series converges by the Ratio Test.