Series

Types of Series

Telescoping Series

Formula: $\sum_{n=1}^{\infty}(a_n - a_{n+1})$

Converges if: partial sums S_n have finite limit as $n \to \infty$

How to solve: Use partial fractions to explicitly write terms, simplify partial sums, take limit as $n \to \infty$.

Geometric Series

Formula: $\sum_{n=0}^{\infty} ar^{n-1}$

Converges if: $|r| < 1$

Diverges if: $|r| \geq 1$

Sum: $\frac{a}{1-r}$

p-Series

Formula: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

Converges if: $p > 1$

Diverges if: $p \leq 1$

Alternating Series

Formula: $\sum_{n=0}^{\infty}(-1)^{n+1}a_n$

Converges if: $a_n > 0$, decreasing, $\lim_{n\to\infty} a_n = 0$

Power Series

Formula: $\sum_{n=0}^{\infty} c_n(x-a)^n$

Converges on interval: $|x-a| < R$

Taylor Series

Formula: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Interval of convergence: via ratio/root test

Convergence Tests

Test for Divergence

Conditions: Any series $\sum a_n$

Diverges if: $\lim_{n\to\infty} a_n \neq 0$.

Integral Test

Conditions: positive, continuous, decreasing

Converges if: $\int_1^{\infty} f(x)\,dx$ converges

Direct Comparison Test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

If $0 \leq b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Limit Comparison Test

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $0 < c < \infty$, then either both series converge or both diverge.

Ratio Test

Compute: $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

Converges if: $L < 1$,

Diverges if: $L > 1$,

Inconclusive: $L = 1$

Root Test

Compute: $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

Converges if: $L < 1$,

Diverges if: $L > 1$,

Inconclusive if: $L = 1$

Alternating Series Test

Form: $\sum(-1)^n a_n$

Converges if: $a_n > 0$, decreasing, $\lim_{n\to\infty} a_n = 0$

Absolute Convergence

Check: $\sum|a_n|$

If convergent: series is absolutely convergent

Error Estimates / Remainders

Alternating Series Remainder

Formula: $|R_n| \leq a_{n+1}$

Integral Test Remainder

Estimate: $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$

(for f positive, continuous, and decreasing)

Taylor Polynomial Remainder

Formula: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$,

where c is between a and x

Power Series Approximation Error

Use Taylor remainder formula