Quiz 9 Outline

Format. This quiz has **3 multiple-choice** questions and **1 free-response** question.

1. (2 points) Determine whether a geometric series converges or diverges.

Example: Determine the value of the following series, if it converges.

$$\sum_{n=0}^{\infty} \frac{5}{6^{n+1}}$$

A. Does not converge
B. 5
C. 1
D. 30
E. $\frac{5}{6}$

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2. (2 points) Use the p-series test to determine whether a series converges or diverges.

Example: Exactly one of the following series converges. Determine which.

A. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$
B. $\sum_{n=1}^{\infty} \frac{1}{n^{-1}}$
C. $\sum_{n=1}^{\infty} \frac{1}{n^{5}}$
D. $\sum_{n=1}^{\infty} \sqrt{n}$

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3. (2 points) Use the Direct Comparison Test or the Limit Comparison Test to determine whether a series converges or diverges.

Example: Consider the series:

$$\sum_{n=4}^{\infty} \frac{5}{3^n - n^3}$$

This series (circle one correct answer):

A. Converges by direct comparison with $b_n = \frac{5}{3^n}$.
B. Converges by limit comparison with $b_n = \frac{1}{3^n}$.
C. Converges by limit comparison with $b_n = \frac{1}{n^3}$.
D. Converges because it is a geometric series with $r = 1/3$.
E. Diverges by the divergence test.

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4. (4 points) Use the Integral Test to determine whether a series converges or diverges.

Example: Determine whether the following series converges or diverges. Fully justify your answer.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$$

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Solutions

Solution: Rewrite the series as

$$\sum_{n=0}^{\infty} \frac{5}{6^{n+1}} = \sum_{n=0}^{\infty} \frac{5}{6}\left(\frac{1}{6}\right)^n.$$

This is a geometric series with

$$a = \frac{5}{6} \qquad \text{and} \qquad r = \frac{1}{6}.$$

Since

$$|r| = \frac{1}{6} < 1,$$

the series converges. Its sum is

$$\sum_{n=0}^{\infty} \frac{5}{6}\left(\frac{1}{6}\right)^n = \frac{a}{1-r} = \frac{\frac{5}{6}}{1-\frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

So the correct choice is $\boxed{\text{C}}$.

Solution: We check each series using the p-series test.

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = \frac{1}{2} < 1$, so it diverges.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n$, whose terms do not even go to 0, so it diverges.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ is a p-series with $p = 5 > 1$, so it converges.
- $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} = \sum_{n=1}^{\infty} n^{1/2}$, whose terms do not go to 0, so it diverges.

Therefore, the only convergent series is $\boxed{\text{C}}$.

Solution:

1. **Introduction:** Let
$$a_n = \frac{5}{3^n - n^3} \qquad \text{and} \qquad b_n = \frac{1}{3^n}.$$
For $n \geq 4$, both $a_n > 0$ and $b_n > 0$, so the Limit Comparison Test applies.

2. **Apply L.C.T.:**
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5}{3^n - n^3}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{5 \cdot 3^n}{3^n - n^3} = \lim_{n\to\infty} \frac{5}{1 - \frac{n^3}{3^n}} = 5.$$
Since this limit is a positive finite number, $\sum a_n$ and $\sum b_n$ either both converge or both diverge. Now
$$\sum_{n=4}^{\infty} \frac{1}{3^n}$$
is a geometric series with ratio
$$r = \frac{1}{3},$$
and
$$|r| = \frac{1}{3} < 1,$$
so it converges.

3. **Conclusion:** Since
$$\lim_{n\to\infty} \frac{a_n}{b_n} = 5 > 0$$
and $\sum b_n$ converges, it follows by the Limit Comparison Test that
$$\sum_{n=4}^{\infty} \frac{5}{3^n - n^3}$$
converges.
Therefore, the correct choice is $\boxed{\text{B}}$.

Solution:

1. **Introduction:** Let

$$f(x) = \frac{1}{x(\ln x)^2}, \qquad x \geq 2.$$

   Then $f(x)$ is positive and continuous for $x \geq 2$. Also,

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3}.$$

   Since $x \geq 2$ implies $\ln x > 0$, we have $f'(x) < 0$ for $x \geq 2$. Thus f is decreasing on $[2, \infty)$, so the Integral Test applies.

2. **Apply the Integral Test:**

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx.$$

   Let

$$u = \ln x \qquad \Rightarrow \qquad du = \frac{1}{x}\, dx.$$

   Then

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 2}^\infty \frac{1}{u^2}\, du = \int_{\ln 2}^\infty u^{-2}\, du = \left[-u^{-1}\right]_{\ln 2}^\infty.$$

   Therefore,

$$\left[-u^{-1}\right]_{\ln 2}^\infty = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

   This is finite, so the improper integral converges.

3. **Conclusion:** Since

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx$$

   converges, the series

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$$

   converges by the Integral Test.