Quiz 8 Outline

Format. This quiz has **3 multiple-choice** questions and **1 free-response** question.

1. (2 points) Does the given sequence converge or diverge?

Example: For the following sequence $\{a_n\}_{n=1}^{\infty}$ below, circle the true statement.

$$a_n = \cos\left(\frac{\pi}{2^n}\right)$$

A. The sequence converges to 0.
B. The sequence converges, but not to 0.
C. The sequence diverges to ∞.
D. The sequence diverges, but not to ∞.

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2. (2 points) Given the partial sums of a series, what can you conclude?

Example: Suppose that a series $\sum_{k=1}^{\infty} a_k$ has partial sums given by

$$S_n = \frac{n+5}{2n+1}.$$

Which of the following is true?

A. $\sum_{k=1}^{\infty} a_k$ diverges by the divergence test.
B. $\sum_{k=1}^{\infty} a_k$ converges by the divergence test.
C. $\sum_{k=1}^{\infty} a_k$ converges to $\frac{1}{2}$.
D. $\lim_{k\to\infty} a_k = \frac{1}{2}$.
E. We cannot conclude anything about the series $\sum_{k=1}^{\infty} a_k$.

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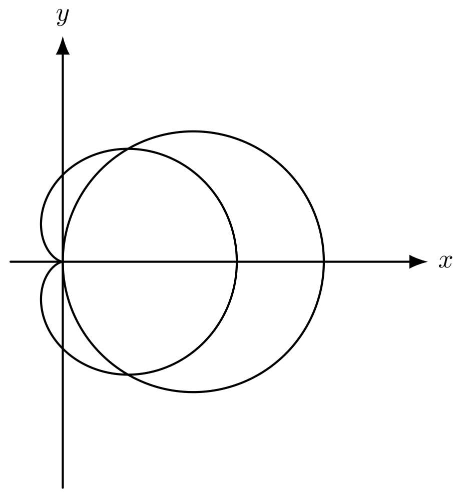
3. (2 points) Given a series, what can you conclude?

Example: Consider the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$. Which of the following statements is true?

(A) The series converges to 0.
(B) The series converges to 1.
(C) The series converges to a value other than 0 or 1.
(D) The series diverges because the sequence of terms approaches 0.
(E) The series diverges because the sequence of terms approaches 1.

4. (4 points) **Fully compute** the area integral of a polar region, where you have to integrate $\cos^2(\theta)$ or $\sin^2(\theta)$.

Example: Find the area of the region in the plane that lies *inside* the polar curve $r = 3\cos\theta$ and *outside* the polar curve $r = 1 + \cos\theta$.



Solutions

1.

Solution: Let $a_n = \cos\left(\frac{\pi}{2^n}\right)$. Since $2^n \to \infty$, we have

$$\frac{\pi}{2^n} \to 0.$$

By continuity of cosine,

$$\lim_{n\to\infty} a_n = \cos(0) = 1.$$

Therefore the sequence *converges, but not to* 0. (Correct choice: **B**.)

2.

Solution: If S_n denotes the nth partial sum of $\sum_{k=1}^{\infty} a_k$, then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \lim_{n\to\infty} S_n \text{ exists,}$$

and in that case the sum equals $\lim_{n\to\infty} S_n$. Here

$$S_n = \frac{n+5}{2n+1} \qquad \Longrightarrow \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{2+\frac{1}{n}} = \frac{1}{2}.$$

So $\sum_{k=1}^{\infty} a_k$ converges to $\frac{1}{2}$. (Correct choice: **C**.)

3.

Solution: Let $a_n = \cos\left(\frac{1}{n}\right)$. Since $\frac{1}{n} \to 0$, we get

$$\lim_{n\to\infty} a_n = \cos(0) = 1 \neq 0.$$

Because the terms of the series do *not* approach 0, the series $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges. (Correct choice: **E**.)

4.

Solution: The curves intersect where

$$3\cos\theta = 1 + \cos\theta \quad\Longrightarrow\quad 2\cos\theta = 1 \quad\Longrightarrow\quad \cos\theta = \frac{1}{2},$$

so

$$\theta = \pm\frac{\pi}{3}.$$

The desired area is

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\Big((3\cos\theta)^2 - (1+\cos\theta)^2\Big)\,d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\Big((3\cos\theta)^2 - (1+\cos\theta)^2\Big)\,d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\big(8\cos^2\theta - 2\cos\theta - 1\big)\,d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\big(3 + 4\cos(2\theta) - 2\cos\theta\big)\,d\theta \\ &= \frac{1}{2}\Big[\,3\theta + 2\sin(2\theta) - 2\sin\theta\,\Big]_{-\pi/3}^{\pi/3} \\ &= \frac{1}{2}\big(\pi - (-\pi)\big) \\ &= \pi. \end{aligned}$$