Quiz 7 Outline

Format. This quiz has **2 multiple-choice** questions, **1 short-answer** question, and **1 free-response** question.

1. (2 points) Compute the equation of the tangent line to a parametric curve at a point (x, y), where you have to solve for t first.

Example: Consider the parametric curve

$$x = t^2, \qquad y = \sqrt{t}, \qquad 0 \le t \le 4.$$

What is the equation of the line tangent to the curve at the point $(1, 1)$?

A. $y = 4x - 3$

B. $y = -\frac{1}{4}x + \frac{5}{4}$

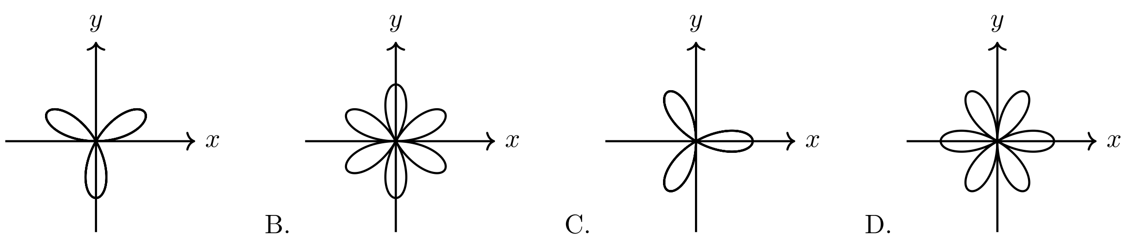
C. $y = \frac{1}{4}x + \frac{3}{4}$

D. $y = x$

E. $y = \frac{1}{2}x + \frac{1}{2}$

2. (2 points) Choose the correct polar graph.

Example: Which of the following graphs represents the polar function $r = \cos(3\theta)$?



A. B. C. D.

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3. (2 points) Set up an integral for the arc length of a parametric curve.

Example: Set up an integral for the arc length of the parametric curve given by

$$\begin{cases} x(t) = \cos(2t), \\ y(t) = t^2, \end{cases} \qquad 0 \le t \le 2\pi.$$

You do not have to compute the integral.

$L =$

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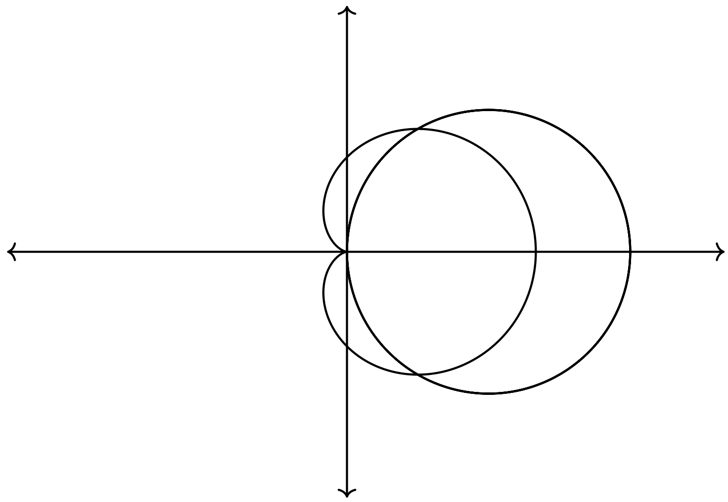
4. (4 points) Set up an integral to compute the area of a polar region. **You must clearly show work for finding the angles of intersection.**

Example Find the area of the region that lies inside the curve $r = 3\cos\theta$ and outside the curve $r = 1 + \cos\theta$.



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Solutions

1. We are given $x = t^2$ and $y = \sqrt{t}$. At the point $(1,1)$,

$$t^2 = 1 \Rightarrow t = 1 \quad (\text{since } 0 \le t \le 4).$$

Differentiate:

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{t}}, \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

At $t = 1$,

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{2} = \frac{1}{4}.$$

Tangent line at $(1,1)$:

$$y - 1 = \frac{1}{4}(x-1) \quad \Longrightarrow \quad y = \frac{1}{4}x + \frac{3}{4}.$$

So the correct choice is $\boxed{y = \frac{1}{4}x + \frac{3}{4}}$.

2. For $r = \cos(3\theta)$, since 3 is odd, the graph is a *3-petal rose*. Also,

$$r(0) = \cos(0) = 1,$$

so one petal is centered on the positive x-axis. This matches option $\boxed{\text{C}}$.

3. Arc length for a parametric curve $(x(t), y(t))$ on $a \le t \le b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Here $x(t) = \cos(2t)$ and $y(t) = t^2$ on $0 \le t \le 2\pi$, so

$$\frac{dx}{dt} = -2\sin(2t), \qquad \frac{dy}{dt} = 2t.$$

Therefore,

$$\boxed{L = \int_0^{2\pi} \sqrt{\left(-2\sin(2t)\right)^2 + (2t)^2}\, dt = \int_0^{2\pi} \sqrt{4\sin^2(2t) + 4t^2}\, dt.}$$

4. The intersection angles occur when the radii are equal:

$$3\cos\theta = 1 + \cos\theta \;\Rightarrow\; 2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \pm\frac{\pi}{3}.$$

The area is

$$\boxed{A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta.}$$