Quiz 4 Outline

Format. This quiz has **3 multiple-choice** questions and **1 free-response** question.

1. (2 points) A u-substitution that requires rewriting the leftover x-factors in terms of u.

Example: Evaluate the definite integral

$$\int_1^2 x\sqrt{x-1}\,dx.$$

A. $\frac{2}{3}$ B. $\frac{16}{15}$ C. $\frac{2}{5}$ D. $\frac{8}{15}$ E. $\frac{4}{3}$

2. (2 points) Determine whether an improper integral with a vertical asymptote inside the interval converges or diverges, and evaluate it if it converges.

Example: Determine whether the improper integral converges or diverges. If it converges, find its value.

$$\int_{-2}^{2} \frac{1}{x^3}\,dx$$

A. It converges to 0.
B. It converges to $\frac{1}{4}$.
C. It converges to $-\frac{1}{4}$.
D. It diverges.
E. There is not enough information to decide.

3. (2 points) Write a definite integral for the volume of a solid when the cross-sectional area is known.

Example: Let R be the region between $y = f(x)$ and $y = g(x)$ on $0 \le x \le 2$, where

$$f(x) = x^2 + 1, \qquad g(x) = x.$$

A solid has base R, and cross-sections perpendicular to the x-axis are squares. Which definite integral represents the volume of the solid?

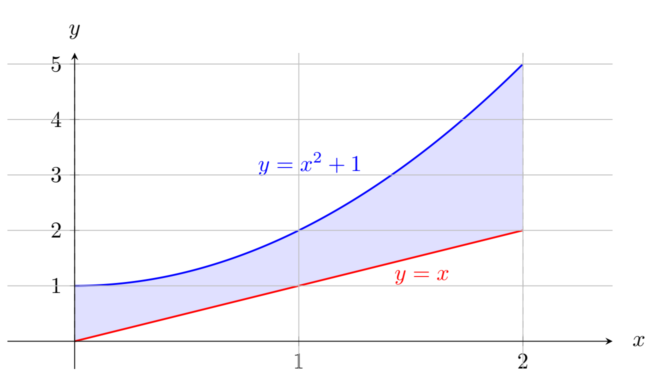
A. $\int_0^2 \left[(x^2+1) - x\right]^2 dx$

B. $\int_0^2 \left[(x^2+1) - x\right] dx$

C. $\int_0^2 \left[(x^2+1) - x^2\right]^2 dx$

D. $\int_0^2 \left[(x^2+1) - x\right]^3 dx$

E. $\int_2^0 \left[(x^2+1) - x\right]^2 dx$



4. (4 points) Find an explicit solution to a separable differential equation, using the initial condition to choose the correct solution branch.

Example Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{\sin x}{y}, \qquad y\left(\frac{\pi}{2}\right) = 2.$$

1. Evaluate $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$.

Solution: Let $u = x - 1$. Then $x = u + 1$ and $du = dx$. Update the bounds:

$$x = 1 \Rightarrow u = 0, \qquad x = 2 \Rightarrow u = 1.$$

So

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 \left(u^{3/2} + u^{1/2}\right)\,du = \left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}.$$

2. Determine whether $\displaystyle\int_{-2}^2 \frac{1}{x^3}\,dx$ converges or diverges.

Solution: The integrand has a vertical asymptote at $x = 0$, which lies inside $[-2, 2]$. Split the integral:

$$\int_{-2}^2 \frac{1}{x^3}\,dx = \int_{-2}^0 \frac{1}{x^3}\,dx + \int_0^2 \frac{1}{x^3}\,dx.$$

Use $\displaystyle\int x^{-3}\,dx = -\frac{1}{2x^2} + C$.

Right side:

$$\int_0^2 \frac{1}{x^3}\,dx = \lim_{b\to 0^+} \int_b^2 x^{-3}\,dx = \lim_{b\to 0^+} \left[-\frac{1}{2x^2}\right]_b^2 = \lim_{b\to 0^+} \left(-\frac{1}{8} + \frac{1}{2b^2}\right) = \infty.$$

Since at least one of the required limits diverges, the improper integral *diverges*.

3. A solid has base R between $f(x) = x^2 + 1$ and $g(x) = x$ on $0 \le x \le 2$, with square cross-sections perpendicular to the x-axis. Which integral gives the volume?

Solution: For cross-sections perpendicular to the x-axis, the side length of each square is

$$s(x) = (\text{top}) - (\text{bottom}) = f(x) - g(x) = (x^2 + 1) - x.$$

So the cross-sectional area is

$$A(x) = s(x)^2 = \left[(x^2 + 1) - x\right]^2.$$

Thus the volume is

$$V = \int_0^2 A(x)\,dx = \int_0^2 \left[(x^2 + 1) - x\right]^2\,dx,$$

which matches choice $\boxed{\text{A}}$.

4. Solve $\dfrac{dy}{dx} = \dfrac{\sin x}{y}$, with $y\left(\frac{\pi}{2}\right) = 2$.

Solution: Separate variables:

$$y\,dy = \sin x\,dx.$$

Integrate:

$$\int y\,dy = \int \sin x\,dx \quad \Rightarrow \quad \frac{1}{2}y^2 = -\cos x + C.$$

Multiply by 2:

$$y^2 = -2\cos x + C_1.$$

Use the initial condition $y\left(\frac{\pi}{2}\right) = 2$. Since $\cos\left(\frac{\pi}{2}\right) = 0$,

$$(2)^2 = -2 \cdot 0 + C_1 \Rightarrow C_1 = 4.$$

So

$$y^2 = 4 - 2\cos x.$$

Taking square roots gives two branches:

$$y = \pm\sqrt{4 - 2\cos x}.$$

Because $y\left(\frac{\pi}{2}\right) = 2 > 0$, we choose the positive branch:

$$\boxed{y = \sqrt{4 - 2\cos x}}.$$