Quiz 2 Outline

Format. This quiz has **3 multiple-choice** questions and **1 free-response** question.

1. (2 points) Use u-substitution to rewrite an integral of the form $\int_a^b \sin^m x \cos^n x \, dx$.

Example: Which of the following is the result of a valid u-substitution for the following integral?

$$\int_0^{\pi/3} \sin^4(x)\cos^3(x)\,dx$$

A. $\displaystyle\int_0^{\sqrt{3}/2} u^4\left(1-u^2\right)du$

B. $\displaystyle\int_0^{\sqrt{3}/2} u^4\left(1-u^2\right)^3du$

C. $\displaystyle\int_0^{1/2} u^3\left(1-u^2\right)^2du$

D. $\displaystyle\int_0^{1/2} u^3\left(1-u^2\right)^4du$

Fall 2022 Exam 1 #2

2. (2 points) Use u-substitution to rewrite an integral of the form $\int_a^b \tan^m x \sec^n x \, dx$.

Example: Which of the following is the result of a valid u-substitution for the following integral?

$$\int_0^{\pi/4} \sec^6(x)\tan^3(x)\,dx$$

A. $\displaystyle\int_0^1 (1+u^2)^2u^3\,du.$

B. $\displaystyle\int_0^1 (1+u^2)^6\,u^2\,du$

C. $\displaystyle\int_0^1 (1-u^2)^3\,u^2\,du$

D. $\displaystyle\int_0^{\sqrt{2}} u^3\,(u^2-1)^2\,du$

E. $\displaystyle\int_0^{\sqrt{2}} u^3\,(u^2+1)^2\,du$

3. (2 points) Determine the correct trig substitution.

Example: Which substitution is appropriate for computing the following integral?

$$\int \frac{\sqrt{x^2-9}}{x^2}\,dx$$

A. $x = 3\sin\theta$

B. $x = 3\sec\theta$

C. $x = 3\tan\theta$

D. $x = 9\sin\theta$

E. $x = 9\sec\theta$

F. $x = 9\tan\theta$

Fall 2022 Exam 1 #5

4. (4 points) The free response problem will be the following question.

Example: Compute the following indefinite integral.

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$$

Your answer will be graded on the quality of your setup and work leading to the correct final antiderivative.

Spring 2023 Exam 1 #9

1. Which of the following is the result of a valid u-substitution for the following integral?

$$\int_0^{\pi/3} \sin^4(x)\cos^3(x)\,dx$$

A. $\displaystyle\int_0^{\sqrt{3}/2} u^4\left(1-u^2\right)du$

B. $\displaystyle\int_0^{\sqrt{3}/2} u^4\left(1-u^2\right)^3 du$

C. $\displaystyle\int_0^{1/2} u^3\left(1-u^2\right)^2 du$

D. $\displaystyle\int_0^{1/2} u^3\left(1-u^2\right)^4 du$

Solution: Use $u = \sin x$, so $du = \cos x\,dx$. Since $\cos^3 x = \cos^2 x \cdot \cos x = (1-\sin^2 x)\cos x$,

$$\int_0^{\pi/3} \sin^4 x\cos^3 x\,dx = \int_0^{\pi/3} \sin^4 x\left(1-\sin^2 x\right)\cos x\,dx.$$

Change bounds: $x = 0 \Rightarrow u = 0$, $x = \pi/3 \Rightarrow u = \sin(\pi/3) = \sqrt{3}/2$. Thus

$$\int_0^{\pi/3} \sin^4 x\cos^3 x\,dx = \int_0^{\sqrt{3}/2} u^4(1-u^2)\,du,$$

which is choice **(A)**.

2. Which of the following is the result of a valid u-substitution for the following integral?

$$\int_0^{\pi/4} \sec^6(x)\tan^3(x)\,dx$$

A. $\displaystyle\int_0^1 (1+u^2)^2 u^3\,du.$

B. $\displaystyle\int_0^1 (1+u^2)^6\,u^2\,du$

C. $\displaystyle\int_0^1 (1-u^2)^3\,u^2\,du$

D. $\displaystyle\int_0^{\sqrt{2}} u^3\,(u^2-1)^2\,du$

E. $\displaystyle\int_0^{\sqrt{2}} u^3\,(u^2+1)^2\,du$

Solution: Use $u = \tan x$, so $du = \sec^2 x\,dx$. Write

$$\sec^6 x\tan^3 x = \sec^4 x\ \tan^3 x\ \sec^2 x = (\sec^2 x)^2\tan^3 x\ \sec^2 x = (1+\tan^2 x)^2\tan^3 x\ \sec^2 x.$$

Then

$$\int_0^{\pi/4} \sec^6 x\tan^3 x\,dx = \int_0^{\pi/4} (1+\tan^2 x)^2\tan^3 x\,(\sec^2 x\,dx) = \int (1+u^2)^2 u^3\,du,$$

with bounds $x = 0 \Rightarrow u = 0$, $x = \pi/4 \Rightarrow u = 1$. So the rewritten integral is

$$\int_0^1 (1+u^2)^2 u^3\,du.$$

which is choice **(A)**.

3. Which substitution is appropriate for computing the following integral?

$$\int \frac{\sqrt{x^2 - 9}}{x^2}\, dx$$

A. $x = 3\sin\theta$

B. $x = 3\sec\theta$

C. $x = 3\tan\theta$

D. $x = 9\sin\theta$

E. $x = 9\sec\theta$

F. $x = 9\tan\theta$

Solution: Since $\sqrt{x^2 - 9} = \sqrt{x^2 - 3^2}$ has the form $\sqrt{x^2 - a^2}$, the standard choice is

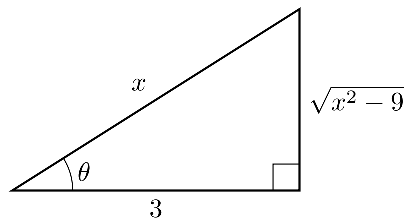
$$x = a\sec\theta \quad \Rightarrow \quad x = 3\sec\theta,$$

which is choice **(B)**.

4. Compute the following indefinite integral.

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$$

Solution: Since $\sqrt{x^2-9} = \sqrt{x^2-3^2}$, use the trig substitution $x = 3\sec\theta$. Then $\sec\theta = \frac{x}{3}$, so a right triangle may be drawn with adjacent 3, hypotenuse x, and opposite $\sqrt{x^2-9}$:



With $x = 3\sec\theta$,

$$dx = 3\sec\theta\tan\theta\,d\theta, \qquad \sqrt{x^2-9} = 3\tan\theta, \qquad x^2 = 9\sec^2\theta.$$

Therefore,

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-9}}\,dx &= \int \frac{1}{(9\sec^2\theta)(3\tan\theta)}\left(3\sec\theta\tan\theta\,d\theta\right)\\ &= \frac{1}{9}\int \cos\theta\,d\theta\\ &= \frac{1}{9}\sin\theta + C.\end{aligned}$$

From the triangle,

$$\sin\theta = \frac{\sqrt{x^2-9}}{x}.$$

Hence,

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \frac{\sqrt{x^2-9}}{9x} + C.$$