

1. (1 point) Which of the following is the Taylor series for

$$f(x) = x^2 \sin(5x^3)$$

centered at  $a = 0$ ?

(a)  $\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n+1)!} x^{6n+5}$

(b)  $\sum_{n=0}^{\infty} 5^n (3n+2)! x^{3n+2}$

(c)  $\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{(6n+5)!} x^{6n+5}$

(d)  $\sum_{n=0}^{\infty} (-1)^n (2n)! 5^{2n} x^{6n+1}$

(e)  $\sum_{n=0}^{\infty} \frac{x^{3n+2}}{(2n+1)!}$

**Solution:** Recall that

$$\sin u = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{(2n+1)!}.$$

Substitute  $u = 5x^3$ :

$$\sin(5x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (5x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n+1)!} x^{6n+3}.$$

Now multiply by  $x^2$ :

$$x^2 \sin(5x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n+1)!} x^{6n+5}.$$

So the correct answer is **(a)**.

2. (1 point) Calculate the value of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n \cdot 3^n}.$$

(a)  $\frac{-2/3}{1 - 2/3} = -2$

(b)  $\ln\left(\frac{1}{3}\right)$

(c)  $\arctan\left(\frac{2}{3}\right)$

(d)  $-\ln\left(\frac{2}{3}\right)$

(e)  $\sin\left(\frac{2}{3}\right)$

**Solution:**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2/3)^n}{n} = \ln\left(1 - \frac{2}{3}\right) = \ln\left(\frac{1}{3}\right).$$

3. (1 point) Find a power series representation of

$$\int \frac{x^2}{1-x^3} dx.$$

(a)  $C + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{3n+2}$

(b)  $C + \sum_{n=0}^{\infty} \frac{x^{3n+3}}{3n+3}$

(c)  $C + \sum_{n=0}^{\infty} \frac{x^{n+3}}{n+3}$

(d)  $C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{3n+3}$

(e)  $C + \sum_{n=0}^{\infty} \frac{x^{3n}}{3n}$

**Solution:** Use the geometric series formula

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \quad |r| < 1.$$

With  $r = x^3$ ,

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}.$$

Multiply by  $x^2$ :

$$\frac{x^2}{1-x^3} = \sum_{n=0}^{\infty} x^{3n+2}.$$

Now integrate term-by-term:

$$\int \frac{x^2}{1-x^3} dx = \int \sum_{n=0}^{\infty} x^{3n+2} dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+3}}{3n+3}.$$

So the correct answer is **(b)**.

4. (4 points) Let  $f(x) = x \ln(x)$ . Find the Taylor series for  $f(x)$  centered at  $a = 3$ .

**Solution:**

The Taylor series for  $f(x)$  centered at  $a = 3$  is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} (x-3)^n.$$

so we need to compute  $f^{(n)}(3)$  and plug these values in. We begin by computing several derivatives:

$$f(x) = x \ln x, \quad f'(x) = \ln x + 1, \quad f''(x) = \frac{1}{x}, \quad f'''(x) = -\frac{1}{x^2}, \quad f^{(4)}(x) = \frac{2}{x^3}.$$

For  $n \geq 2$ , the pattern is

$$f^{(n)}(x) = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}.$$

Now evaluate at  $x = 3$ :

$$f(3) = 3 \ln 3, \quad f'(3) = \ln 3 + 1,$$

and for  $n \geq 2$ ,

$$f^{(n)}(3) = (-1)^{n-2} \frac{(n-2)!}{3^{n-1}}.$$

Substituting the values above gives:

$$\begin{aligned} x \ln(x) &= 3 \ln 3 + (\ln 3 + 1)(x-3) + \sum_{n=2}^{\infty} \frac{(-1)^{n-2} (n-2)!}{n! 3^{n-1}} (x-3)^n \\ &= 3 \ln 3 + (\ln 3 + 1)(x-3) + \sum_{n=2}^{\infty} (-1)^{n-2} \frac{(x-3)^n}{n(n-1)3^{n-1}} \end{aligned}$$

5. (3 points) What aspects of MATH 2300 were most helpful for your learning this semester? Is there anything you wish had been added, changed, or done differently to better support your learning?

**Solution:** MATH 2300 is awesome!