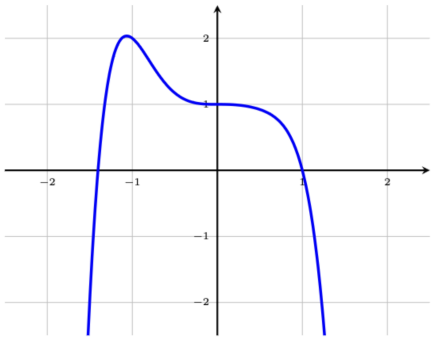
Quiz 12 Outline

Format. This quiz has **3 multiple-choice** questions and **1 free-response** question.

1. (2 points) Determine whether a geometric series converges or diverges.

> **Example:** Determine the value of the following series, if it converges.
>
> $$\sum_{n=0}^{\infty} \frac{5}{6^{n+1}}$$
>
> A. 5 B. 1 C. 30 D. $\frac{5}{6}$ E. Does not converge
>
> *Fall 2022 Exam 3 #2*

2. (2 points) Graphical Taylor Polynomial question.

> **Example.** Suppose a function $f(x)$ has the graph shown below. The 2nd-degree Taylor polynomial for $f(x)$ centered at $a = -1$ is
>
> $$T_2(x) = b(x+1)^2 + c(x+1) + d.$$
>
> Which of the following gives the correct signs of b, c, and d?
>
> A. $b > 0,\ c > 0,\ d > 0$
>
> B. $b > 0,\ c < 0,\ d > 0$
>
> C. $b < 0,\ c > 0,\ d > 0$
>
> D. $b < 0,\ c < 0,\ d > 0$
>
> E. $b < 0,\ c < 0,\ d < 0$
>
> 
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>
> *Homework 11 #4*

3. (2 points) Conceptual question about the radius of convergence.

> **Example.** You are told that the power series
>
> $$\sum_{n=0}^{\infty} c_n (x-2)^n$$
>
> converges at $x = -1$ and diverges at $x = 8$. Which of the following is the **largest interval** that is guaranteed to consist entirely of values of x for which the series converges?
>
> A. $(-4, 8)$
>
> B. $(-1, 5)$
>
> C. $(0, 4)$
>
> D. $(1, 7)$
>
> E. $(-2, 6)$

4. (4 points) Find the interval of convergence of a given power series.

Example. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 3^n}.$$

Solutions

Solution: Write the series as

$$\sum_{n=0}^{\infty} \frac{5}{6^{n+1}} = \frac{5}{6} \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n.$$

This is a geometric series with first term $a = \frac{5}{6}$ and common ratio $r = \frac{1}{6}$. Since $|r| < 1$, the series converges, and its sum is

$$\frac{a}{1-r} = \frac{\frac{5}{6}}{1-\frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

Therefore, the correct answer is **B**.

Solution: For a 2nd-degree Taylor polynomial centered at $a = -1$,

$$T_2(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2}(x+1)^2.$$

Comparing this with

$$T_2(x) = b(x+1)^2 + c(x+1) + d,$$

we have

$$d = f(-1), \qquad c = f'(-1), \qquad b = \frac{f''(-1)}{2}.$$

From the graph:

- At $x = -1$, the function value is above the x-axis, so $f(-1) > 0$. Hence $d > 0$.
- At $x = -1$, the graph is decreasing, so $f'(-1) < 0$. Hence $c < 0$.
- Near $x = -1$, the graph is concave down, so $f''(-1) < 0$. Hence $b < 0$.

Therefore, the correct answer is

$$\boxed{b < 0,\ c < 0,\ d > 0},$$

which is **D**.

Solution: The series is centered at $x = 2$. Since it converges at $x = -1$, the radius of convergence must satisfy

$$R \geq 3.$$

Therefore, the series is guaranteed to converge for every x satisfying

$$|x - 2| < 3,$$

which means

$$-1 < x < 5.$$

So the largest interval we can guarantee lies entirely inside the interval of convergence is

$$\boxed{(-1, 5)}.$$

The fact that the series diverges at $x = 8$ tells us that the radius cannot be larger than 6, but that does not change the guaranteed interval coming from the convergence at $x = -1$. Therefore, the correct answer is **B**.

Solution: Consider

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n \cdot 3^n}.$$

Let

$$a_n = \frac{(x+2)^n}{n \cdot 3^n}.$$

Using the Ratio Test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x+2)^n}\right| = \left|\frac{x+2}{3}\right| \frac{n}{n+1}.$$

Taking the limit as $n \to \infty$,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x+2}{3}\right|.$$

The series converges when $L < 1$, so

$$\left|\frac{x+2}{3}\right| < 1 \quad \Longrightarrow \quad |x+2| < 3.$$

Thus the series converges absolutely for

$$-5 < x < 1.$$

Now test the endpoints:

At $x = -5$:

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is an alternating harmonic series, which converges by the A.S.T. (explain!).

At $x = 1$:

$$\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is the harmonic series, which diverges. Therefore, the interval of convergence is

$$\boxed{[-5,\, 1)}.$$