# Quiz 11 Outline

**Format.** This quiz has **2 multiple-choice** questions, **1 short-answer** question, and **1 free-response** question.

1. (2 points) Apply the Ratio Test.

**Example.** What does the Ratio Test tell us about the following series?

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

A. The Ratio Test tells us that the series converges absolutely.
B. The Ratio Test tells us that the series converges conditionally.
C. The Ratio Test tells us that the series diverges.
D. The Ratio Test is inconclusive.

*Fall 2022 Exam 3 #5*

2. (2 points) Apply the Integral Test Remainder Estimate.

**Example.** Let

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

If $R_5 = S - S_5$ is the remainder after the 5th partial sum, which of the following must be true?

A. $\frac{1}{6} \leq R_5 \leq \frac{1}{5}$
B. $\frac{1}{5} \leq R_5 \leq \frac{1}{4}$
C. $0 \leq R_5 \leq \frac{1}{6}$
D. $R_5 \geq \frac{1}{5}$
E. $R_5 = \frac{1}{6}$

3. (2 points) Apply the Alternating Series Estimation Theorem.

**Example.** Let

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)^2}.$$

If $S_6$ is used to approximate $S$, what is the largest possible value of $|S - S_6|$? Do not simplify your answer.

4. (4 points) Compute a Taylor polynomial and use it to estimate a function value.

**Example.**

(a) Find the 3rd-degree Taylor polynomial centered at $a = 2$ for the function $f(x) = \frac{1}{x}$.

(b) Use your result from part (a) to approximate $\frac{1}{2.1}$. You do not need to simplify your answer.

## Solutions

**Solution:** Let
$$a_n = \frac{\ln(n)}{n^2}.$$
Then
$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(n)}\cdot\frac{n^2}{(n+1)^2}\right) = \left(\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n)}\right)\left(\lim_{n\to\infty}\frac{n^2}{(n+1)^2}\right).$$
Now,
$$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n)} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1,$$
and
$$\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1.$$
Thus, $L = 1$. Since $L = 1$, the Ratio Test is inconclusive. Therefore, the correct answer is **D**.

**Solution:** Here,
$$f(x) = \frac{1}{x^2},$$
which is continuous, positive, and decreasing for $x \geq 1$. By the Integral Test Remainder Estimate,
$$\int_6^\infty \frac{1}{x^2}\,dx \leq R_5 \leq \int_5^\infty \frac{1}{x^2}\,dx.$$
Now,
$$\int_6^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_6^\infty = \frac{1}{6},$$
and
$$\int_5^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_5^\infty = \frac{1}{5}.$$
Therefore,
$$\frac{1}{6} \leq R_5 \leq \frac{1}{5},$$
so the correct answer is **A**.

**Solution:** Let
$$b_n = \frac{1}{(2n+1)^2}.$$
Since $b_n > 0$, $b_n$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, the Alternating Series Estimation Theorem applies. Therefore,
$$|S - S_6| \leq b_7.$$
Since
$$b_7 = \frac{1}{(2\cdot 7+1)^2} = \frac{1}{15^2} = \frac{1}{225},$$
the largest possible error is
$$\boxed{\frac{1}{225}}$$

**Solution:**

(a) We compute the derivatives:

$$f(x) = x^{-1}, \qquad f'(x) = -x^{-2}, \qquad f''(x) = 2x^{-3}, \qquad f'''(x) = -6x^{-4}.$$

Evaluating at $a = 2$,

$$f(2) = \frac{1}{2}, \qquad f'(2) = -\frac{1}{4}, \qquad f''(2) = \frac{1}{4}, \qquad f'''(2) = -\frac{3}{8}.$$

Therefore,

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

becomes

$$T_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3.$$

(b) Use $x = 2.1$:

$$\frac{1}{2.1} \approx T_3(2.1) = \frac{1}{2} - \frac{1}{4}(0.1) + \frac{1}{8}(0.1)^2 - \frac{1}{16}(0.1)^3.$$