Quiz 10 Outline

Format. This quiz has **2 free-response** questions. The main goal of this quiz is to practice writing clear, detailed, and precise solutions. In particular, whenever you use a convergence test, be sure to:

1. State which test you are using and verify that its necessary conditions are satisfied
2. Apply the test clearly
3. State the conclusion that follows from the test

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1. (5 points) Determine whether a given series converges absolutely, converges conditionally, or diverges.

> **Example:** Consider the series
>
> $$\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}.$$
>
> Does the series converge absolutely, converge conditionally, or diverge? Fully justify your answer.
>
> *Spring 2023 Exam 3 #6*

2. (5 points) Use the Ratio Test to determine whether a series converges or diverges.

> **Example:** Apply the ratio test to determine if the following series converges or diverges.
>
> $$\sum_{n=1}^{\infty} \frac{2^n(n!)}{(2n)!}$$
>
> *Spring 2023 Exam 3 #7*

Solutions

1. Consider the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}.$$

Does the series converge absolutely, converge conditionally, or diverge? Fully justify your answer.

Solution: First, check for absolute convergence:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

This is a p-series with $p = \frac{1}{2} < 1$, so it diverges. Therefore, the series does *not* converge absolutely.

Now check whether the original alternating series converges:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$$

Let

$$b_n = \frac{1}{\sqrt{n}}.$$

Then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Also, b_n is decreasing because $\sqrt{n}$ is increasing, so $\frac{1}{\sqrt{n}}$ is decreasing. Therefore, by the Alternating Series Test, the series converges. Since the series converges, but its absolute value series diverges, the series **converges conditionally**.

2. Apply the ratio test to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n(n!)}{(2n)!}$$

Solution: Using the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{(2n+2)!}\cdot\frac{(2n)!}{2^n n!}\right| \\
&= \lim_{n\to\infty}\frac{2^{n+1}(n+1)!(2n)!}{(2n+2)!\,2^n n!} \\
&= \lim_{n\to\infty}\frac{2\big((n+1)\cancel{n!}\big)\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{n!}} \\
&= \lim_{n\to\infty}\frac{2(n+1)}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty}\frac{2n+2}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty}\frac{1}{2n+1} \\
&= 0
\end{aligned}$$

Since $L = 0 < 1$, the series **converges** by the Ratio Test.