Math 2300: Final Exam Practice (Solutions)

1. Evaluate $\int_0^1 xe^{2x}\,dx$.

Solution: Let

$$u = x, \qquad dv = e^{2x}\,dx.$$

Then

$$du = dx, \qquad v = \frac{1}{2}e^{2x}.$$

Thus, by integration by parts,

$$\begin{aligned}\int_0^1 xe^{2x}\,dx &= \frac{x}{2}e^{2x}\Big|_0^1 - \int_0^1 \frac{1}{2}e^{2x}\,dx \\ &= \frac{x}{2}e^{2x}\Big|_0^1 - \frac{1}{4}e^{2x}\Big|_0^1 \\ &= \frac{1}{2}e^2 - \left(\frac{1}{4}e^2 - \frac{1}{4}\right) \\ &= \frac{e^2+1}{4}.\end{aligned}$$

2. Evaluate $\int \frac{x}{\sqrt{1+x^2}}\,dx$.

Solution:

Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$.

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}}\,dx &= \frac{1}{2}\int u^{-1/2}\,du \\ &= \frac{1}{2}\cdot 2u^{1/2} + C \\ &= \sqrt{1+x^2} + C\end{aligned}$$

3. Evaluate $\int \sin^5(x)\cos^2(x)\,dx$.

Solution:

$$\begin{aligned}\int \sin^5(x)\cos^2(x)\,dx &= \int \sin^4(x)\sin(x)\cos^2(x)\,dx\\ &= \int (1-\cos^2 x)^2 \sin x \cos^2 x\,dx\end{aligned}$$

Let $u = \cos x$. Then $du = -\sin x\,dx$.

$$\begin{aligned}\int \sin^5(x)\cos^2(x)\,dx &= -\int (1-u^2)^2 u^2\,du\\ &= -\int (u^2 - 2u^4 + u^6)\,du\\ &= -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C\\ &= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C\end{aligned}$$

4. Evaluate $\int \sec^3(x)\tan^3(x)\,dx$.

Solution:

$$\begin{aligned}\int \sec^3(x)\tan^3(x)\,dx &= \int \sec^3(x)\tan^2(x)\tan(x)\,dx\\ &= \int \sec^3(x)(\sec^2 x - 1)\tan(x)\,dx\\ &= \int \sec^2(x)(\sec^2 x - 1)\big(\sec x \tan x\,dx\big)\end{aligned}$$

Let $u = \sec x$. Then $du = \sec x \tan x\,dx$.

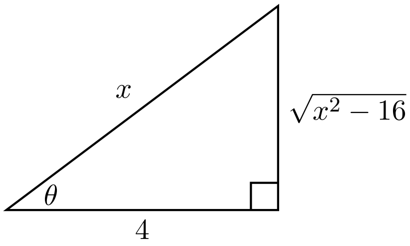
$$\begin{aligned}\int \sec^3(x)\tan^3(x)\,dx &= \int u^2(u^2-1)\,du\\ &= \int (u^4 - u^2)\,du\\ &= \frac{u^5}{5} - \frac{u^3}{3} + C\\ &= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C\end{aligned}$$

5. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$.

Solution: Use the substitution $x = 4\sec\theta$. Then

$$\sec\theta = \frac{x}{4}.$$

So we begin with the reference triangle:



From the substitution,

$$dx = 4\sec\theta\tan\theta\,d\theta, \qquad \sqrt{x^2-16} = \sqrt{16\sec^2\theta - 16} = 4\tan\theta.$$

Now substitute into the integral:

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-16}}\,dx &= \int \frac{1}{(16\sec^2\theta)(4\tan\theta)} \cdot 4\sec\theta\tan\theta\,d\theta \\ &= \frac{1}{16}\int \cos\theta\,d\theta \\ &= \frac{1}{16}\sin\theta + C.\end{aligned}$$

From the triangle,

$$\sin\theta = \frac{\sqrt{x^2-16}}{x}.$$

Therefore,

$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx = \frac{\sqrt{x^2-16}}{16x} + C.$$

6. Evaluate $\int \frac{5x-1}{x^2-2x-3}\,dx$.

Solution: Factor the denominator:

$$x^2-2x-3=(x-3)(x+1).$$

Use partial fractions:

$$\frac{5x-1}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}.$$

Then

$$5x-1=A(x+1)+B(x-3).$$

So

$$\begin{aligned} A+B&=5,\\ A-3B&=-1. \end{aligned}$$

Solving gives

$$A=\frac{7}{2}, \qquad B=\frac{3}{2}.$$

Therefore,

$$\begin{aligned} \int \frac{5x-1}{x^2-2x-3}\,dx &= \int \left(\frac{7/2}{x-3}+\frac{3/2}{x+1}\right)\,dx \\ &= \frac{7}{2}\ln|x-3|+\frac{3}{2}\ln|x+1|+C. \end{aligned}$$

7. The general solution to a differential equation is $y=Ce^{3x}-x-\frac{1}{3}$. Find the particular solution satisfying the initial condition $y(0)=2$.

Solution: Substitute $x=0$ into the general solution:

$$2=Ce^0-0-\frac{1}{3}=C-\frac{1}{3}.$$

So

$$C=\frac{7}{3}.$$

Therefore, the particular solution is

$$y=\frac{7}{3}e^{3x}-x-\frac{1}{3}.$$

8. Solve the differential equation $y' = y^2$, with $y(0) = 1$.

Solution: Separate variables:

$$\frac{dy}{dx} = y^2 \qquad \Longrightarrow \qquad \frac{1}{y^2}\,dy = dx.$$

Integrate both sides:

$$\int y^{-2}\,dy = \int dx$$
$$-\frac{1}{y} = x + C.$$

Solve for y:

$$\frac{1}{y} = -x + C \qquad \Longrightarrow \qquad y = \frac{1}{C - x}.$$

Now use $y(0) = 1$:

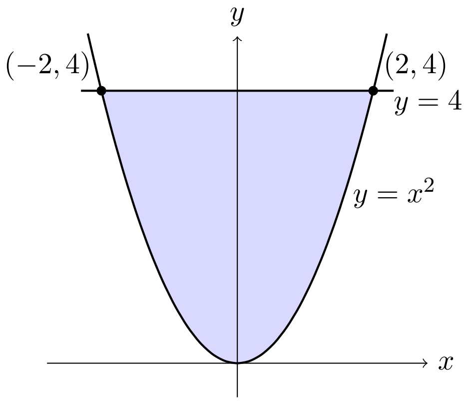
$$1 = \frac{1}{C},$$

so $C = 1$. Thus,

$$y = \frac{1}{1 - x}.$$

9. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 4$ about the x-axis.

Solution: A sketch of the region is shown below.



First find the points of intersection:

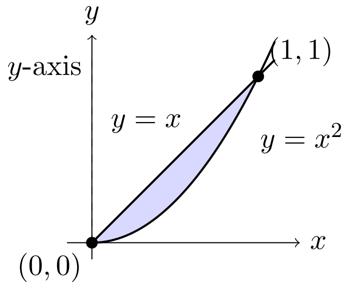
$$x^2 = 4 \qquad \Longrightarrow \qquad x = \pm 2.$$

Using washers, the outer radius is $R = 4$, and the inner radius is $r = x^2$. So the volume is

$$V = \pi \int_{-2}^{2} \left(R^2 - r^2\right)\, dx = \pi \int_{-2}^{2} \left(16 - x^4\right)\, dx = \pi \left[16x - \frac{x^5}{5}\right]_{-2}^{2} = \frac{256\pi}{5}.$$

10. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about the y-axis.

Solution: A sketch of the region is shown below.



First find the points of intersection:

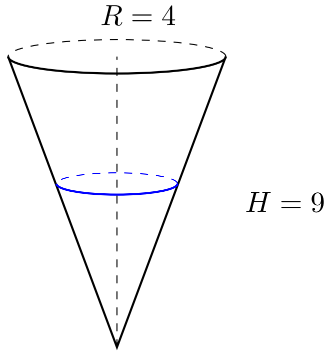
$$x = x^2 \qquad \Rightarrow \qquad x(x-1) = 0, \qquad \Rightarrow \qquad x = 0, 1.$$

Using shells, the radius is $r = x$, and the height is $h = x - x^2$. Thus,

$$V = 2\pi \int_0^1 x(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}.$$

11. A tank has the shape of an inverted circular cone with height 9 m and base radius 4 m. It is filled with water to a height of 5 m (measured upward from the tip). How much work is required to pump all of the water to the top rim of the tank? Set up, but do not evaluate, an integral for the work required. Assume the density of water is 1000 kg/m^3 and take $g = 9.8$ m/s^2.

Solution: Measure height y upward from the tip of the cone. Then the water occupies $0 \le y \le 5$.



Measure height y upward from the tip, so $0 \le y \le 5$. Use

$$W = \int_a^b (\rho g)(\text{area of slice})(\text{distance})\,dy.$$

Here, $\rho = 1000$, $g = 9.8$, and by similar triangles $\frac{r}{y} = \frac{4}{9}$, so $r = \frac{4}{9}y$. Thus

$$\text{area} = \pi r^2 = \pi\left(\frac{4}{9}y\right)^2, \qquad \text{distance} = 9 - y.$$

Therefore,

$$W = \int_0^5 9800\pi\left(\frac{4}{9}y\right)^2(9 - y)\,dy.$$

12. Compute the center of mass of the region bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

Solution: For the region under $y = \sin x$ on $[0, \pi]$,

$$A = \int_0^\pi \sin x \, dx = [-\cos x]_0^\pi = 2.$$

The x-coordinate is

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_0^\pi x \sin x \, dx \\ &= \frac{1}{2} [-x \cos x + \sin x]_0^\pi \\ &= \frac{1}{2}(\pi) = \frac{\pi}{2}. \end{aligned}$$

The y-coordinate is

$$\begin{aligned} \bar{y} &= \frac{1}{2A} \int_0^\pi (\sin x)^2 \, dx \\ &= \frac{1}{4} \int_0^\pi \sin^2 x \, dx \\ &= \frac{1}{4} \int_0^\pi \frac{1 - \cos(2x)}{2} \, dx \\ &= \frac{1}{4} \left[\frac{x}{2} - \frac{\sin(2x)}{4} \right]_0^\pi \\ &= \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}. \end{aligned}$$

Thus, the center of mass is

$$\left(\frac{\pi}{2}, \frac{\pi}{8} \right).$$

13. Compute the equation of the tangent line to the parametric curve $x = \ln(t)$, $y = t^2$ at $t = 1$.

Solution: First find the point:

$$x(1) = \ln(1) = 0, \qquad y(1) = 1^2 = 1.$$

So the point is $(0, 1)$. Now compute the slope:

$$\frac{dx}{dt} = \frac{1}{t}, \qquad \frac{dy}{dt} = 2t.$$

Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2.$$

At $t = 1$,

$$\frac{dy}{dx} = 2.$$

Therefore, the tangent line is

$$y - 1 = 2(x - 0) \quad \Leftrightarrow \quad y = 2x + 1.$$

14. For the parametric curve

$$x = t^3 - 3t, \qquad y = t^4 - 4t^2,$$

find all values of t at which the curve has a horizontal tangent line.

Solution: A horizontal tangent occurs when $\frac{dy}{dx} = 0$, so we need $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

Differentiate:

$$\frac{dx}{dt} = 3t^2 - 3 = 3(t^2 - 1), \qquad \frac{dy}{dt} = 4t^3 - 8t = 4t(t^2 - 2).$$

Set $\frac{dy}{dt} = 0$:

$$4t(t^2 - 2) = 0 \qquad \Longrightarrow \qquad t = 0,\ \pm\sqrt{2}.$$

Now check that $\frac{dx}{dt} \neq 0$ at these values:

$$\left.\frac{dx}{dt}\right|_{t=0} = -3 \neq 0, \qquad \left.\frac{dx}{dt}\right|_{t=\pm\sqrt{2}} = 3 \neq 0.$$

Therefore, the curve has horizontal tangent lines at

$$t = 0,\ \pm\sqrt{2}.$$

15. Find the arc length of the curve $x = t^2$, $y = t^3$ on the interval $0 \le t \le 1$.

Solution: The arc length is

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Differentiate:

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 3t^2.$$

So

$$\begin{aligned} L &= \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt \\ &= \int_0^1 \sqrt{4t^2 + 9t^4}\, dt \\ &= \int_0^1 t\sqrt{4 + 9t^2}\, dt. \end{aligned}$$

Let

$$u = 4 + 9t^2, \qquad du = 18t\, dt.$$

Then

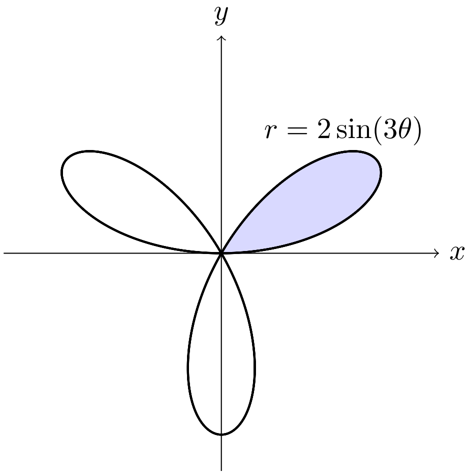
$$\begin{aligned} L &= \frac{1}{18}\int_4^{13} u^{1/2}\, du \\ &= \frac{1}{18} \cdot \frac{2}{3} u^{3/2}\Big|_4^{13} \\ &= \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) \\ &= \frac{1}{27}\left(13\sqrt{13} - 8\right). \end{aligned}$$

Therefore,

$$L = \frac{13\sqrt{13} - 8}{27}.$$

16. Compute the area inside one petal of the rose $r = 2\sin(3\theta)$.

Solution: A sketch of the curve is shown below.



One petal is traced out for $0 \leq \theta \leq \frac{\pi}{3}$. Thus,

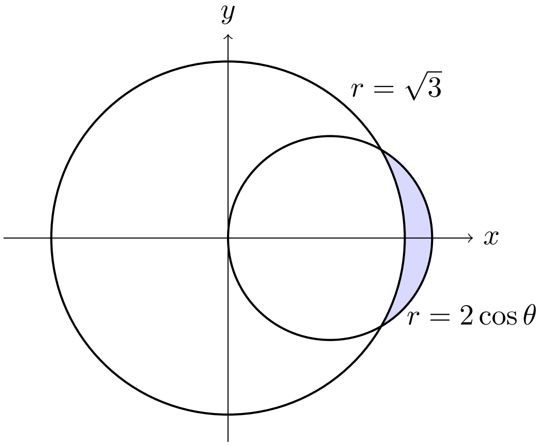
$$A = \frac{1}{2}\int_0^{\pi/3} \big(2\sin(3\theta)\big)^2 \, d\theta = 2\int_0^{\pi/3} \sin^2(3\theta)\, d\theta.$$

Let $u = 3\theta$, so $du = 3\, d\theta$. Then

$$A = \frac{2}{3}\int_0^{\pi} \sin^2 u \, du = \frac{\pi}{3}.$$

17. Find the area of the polar region that lies outside $r = \sqrt{3}$ and inside $r = 2\cos\theta$.

Solution: A sketch of the region is shown below.



First find the points of intersection:

$$\sqrt{3} = 2\cos\theta \qquad \Longrightarrow \qquad \cos\theta = \frac{\sqrt{3}}{2} \qquad \Longrightarrow \qquad \theta = \pm\frac{\pi}{6}.$$

The desired region lies between the curves for $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$, with outer radius $r = 2\cos\theta$ and inner radius $r = \sqrt{3}$. Thus,

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left((2\cos\theta)^2 - (\sqrt{3})^2\right)\, d\theta \\
&= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(4\cos^2\theta - 3\right)\, d\theta \\
&= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(4\left[\frac{1+\cos(2\theta)}{2}\right] - 3\right)\, d\theta \\
&= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \left(-1 + 2\cos(2\theta)\right)\, d\theta \\
&= \frac{1}{2}\left[-\theta + \sin(2\theta)\right]_{-\pi/6}^{\pi/6} \\
&= \frac{1}{2}\left[\left(-\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)\right] \\
&= \frac{1}{2}\left(-\frac{\pi}{3} + \sqrt{3}\right) \\
&= \frac{\sqrt{3}}{2} - \frac{\pi}{6}.
\end{aligned}$$

18. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3n}-\frac{1}{3(n+4)}\right)$.

Solution: Let

$$s_n=\sum_{k=1}^{n}\left(\frac{1}{3k}-\frac{1}{3(k+4)}\right).$$

Then

$$s_n=\left(\frac{1}{3}-\frac{1}{15}\right)+\left(\frac{1}{6}-\frac{1}{18}\right)+\left(\frac{1}{9}-\frac{1}{21}\right)+\left(\frac{1}{12}-\frac{1}{24}\right)+\cdots+\left(\frac{1}{3n}-\frac{1}{3n+12}\right).$$

Writing the positive and negative terms in order,

$$\begin{aligned} s_n={}&\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\frac{1}{15}+\cdots+\frac{1}{3n}\right)\\ &-\left(\frac{1}{15}+\frac{1}{18}+\frac{1}{21}+\frac{1}{24}+\cdots+\frac{1}{3n}+\frac{1}{3n+3}+\frac{1}{3n+6}+\frac{1}{3n+9}+\frac{1}{3n+12}\right).\end{aligned}$$

So everything from $\frac{1}{15}$ through $\frac{1}{3n}$ cancels, leaving

$$s_n=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}-\frac{1}{3n+3}-\frac{1}{3n+6}-\frac{1}{3n+9}-\frac{1}{3n+12}.$$

Now let $n\to\infty$. The last four terms go to 0, so

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{3n}-\frac{1}{3(n+4)}\right)&=\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}\\ &=\frac{12+6+4+3}{36}\\ &=\frac{25}{36}.\end{aligned}$$

19. Evaluate the series $\displaystyle\sum_{n=3}^{\infty}\frac{2^n}{5^{n+2}}$.

Solution: Rewrite the series as

$$\sum_{n=3}^{\infty}\frac{2^n}{5^{n+2}}=\sum_{n=3}^{\infty}\frac{1}{25}\cdot\left(\frac{2}{5}\right)^n.$$

This is geometric with ratio $r=\frac{2}{5}$, so

$$\begin{aligned}\sum_{n=3}^{\infty}\frac{2^n}{5^{n+2}}&=\frac{1}{25}\cdot\frac{(2/5)^3}{1-2/5}\\ &=\frac{1}{25}\cdot\frac{8/125}{3/5}\\ &=\frac{1}{25}\cdot\frac{8}{125}\cdot\frac{5}{3}\\ &=\frac{8}{1875}.\end{aligned}$$

20. Determine whether the series $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$ converges or diverges.

Solution: Use the Test for Divergence:

$$\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0.$$

Since the terms do not approach 0, the series

$$\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$$

diverges.

21. Determine whether the series $\sum_{n=1}^{\infty} \frac{\arcsin(1/n)}{n^3}$ converges or diverges. Justify your answer.

Solution: Since $0 < \frac{1}{n} \leq 1$ for all $n \geq 1$, we have

$$0 < \arcsin(1/n) \leq \frac{\pi}{2}.$$

Therefore,

$$0 \leq \frac{\arcsin(1/n)}{n^3} \leq \frac{\pi/2}{n^3}.$$

Let $a_n = \frac{\arcsin(1/n)}{n^3}$ and $b_n = \frac{\pi/2}{n^3}$. Both are positive, so the Direct Comparison Test applies. Also,

$$\sum_{n=1}^{\infty} \frac{\pi/2}{n^3} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

This is a constant multiple of a p-series with $p = 3 > 1$, so it converges. Since $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test.

22. Find a power series representation for $x\cos(3x^2)$.

Solution: Start with the Maclaurin series

$$\cos u = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n}}{(2n)!}.$$

Substitute $u = 3x^2$:

$$\cos(3x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(3x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{9^n x^{4n}}{(2n)!}.$$

Now multiply by x:

$$x\cos(3x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{9^n x^{4n+1}}{(2n)!}.$$

23. Evaluate the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{\,2n+1}}$.

Solution: Recall the Taylor series

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \qquad -1 \le x \le 1.$$

Substituting $x = \frac{1}{3}$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{\,2n+1}} = \arctan\left(\frac{1}{3}\right).$$

24. Find a power series representation for $\int \frac{x^2}{1+x^3}\, dt$.

Solution: Interpreting the integral as

$$\int \frac{x^2}{1+x^3}\, dx,$$

start with

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \qquad |x| < 1.$$

Multiplying by x^2,

$$\frac{x^2}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n+2}.$$

Now integrate term-by-term:

$$\begin{aligned} \int \frac{x^2}{1+x^3}\, dx &= \int \sum_{n=0}^{\infty} (-1)^n x^{3n+2}\, dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{3n+3} + C. \end{aligned}$$

Thus,

$$\int \frac{x^2}{1+x^3}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+3}}{3n+3} + C, \qquad |x| < 1.$$

25. Find the Taylor series for $f(x) = x\ln(x)$ centered at $a = 4$.

Solution: Recall that the Taylor series for $f(x)$ centered at $a = 4$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n.$$

We first compute derivatives:

$$f(x) = x\ln x, \qquad f'(x) = \ln x + 1, \qquad f''(x) = \frac{1}{x}, \qquad f'''(x) = -\frac{1}{x^2}, \qquad f^{(4)}(x) = \frac{2}{x^3}.$$

For $n \geq 2$, the derivatives follow the pattern

$$f^{(n)}(x) = \frac{(-1)^n(n-2)!}{x^{n-1}}.$$

Now evaluate at $x = 4$:

$$f(4) = 4\ln 4, \qquad f'(4) = \ln 4 + 1,$$

and for $n \geq 2$,

$$f^{(n)}(4) = \frac{(-1)^n(n-2)!}{4^{n-1}}.$$

Substitute into the Taylor series formula:

$$\begin{aligned} f(x) &= f(4) + f'(4)(x-4) + \sum_{n=2}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n \\ &= 4\ln 4 + (\ln 4 + 1)(x-4) + \sum_{n=2}^{\infty} \frac{(-1)^n(n-2)!}{n!\,4^{n-1}}(x-4)^n \\ &= 4\ln 4 + (\ln 4 + 1)(x-4) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)4^{n-1}}(x-4)^n. \end{aligned}$$

Therefore, the Taylor series is

$$x\ln x = 4\ln 4 + (\ln 4 + 1)(x-4) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)4^{n-1}}(x-4)^n.$$

26. Find the Taylor series for $f(x) = \dfrac{1}{x}$ centered at $a = 3$. Also, determine its interval of convergence.

Solution: Rewrite $f(x)$ in geometric-series form:

$$\frac{1}{x} = \frac{1}{3 + (x-3)} = \frac{1}{3} \cdot \frac{1}{1 + \frac{x-3}{3}}.$$

Now use

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \qquad |r| < 1,$$

with

$$r = -\frac{x-3}{3}.$$

Then

$$\frac{1}{x} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x-3}{3} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n.$$

So the Taylor series is

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n.$$

For convergence,

$$\left| -\frac{x-3}{3} \right| < 1 \qquad \Longrightarrow \qquad |x-3| < 3.$$

So the interval of convergence is

$$(0, 6).$$

27. Use Taylor's Inequality to determine the minimum degree n such that the Taylor polynomial $T_n(x)$ for e^x, centered at $a = 2$, approximates e^3 to within 0.01.

Solution: We want to approximate e^3 using the Taylor polynomial for e^x centered at $a = 2$. We can use Taylor's Inequality on the interval $[1, 3]$. For $f(x) = e^x$, every derivative is e^x. Thus, on $[1, 3]$,

$$|f^{(n+1)}(x)| \leq e^3,$$

so we may take $M = e^3$. Taylor's Inequality gives

$$|R_n(3)| \leq \frac{M}{(n+1)!}|3-2|^{n+1} = \frac{e^3}{(n+1)!}.$$

We want this error bound to be less than 0.01, so we need

$$\frac{e^3}{(n+1)!} < 0.01.$$

Now check values of n:

$$n = 5: \quad \frac{e^3}{6!} = \frac{e^3}{720} > \frac{8}{720} = \frac{1}{90} > 0.01$$

$$n = 6: \quad \frac{e^3}{7!} = \frac{e^3}{5040} < \frac{27}{5040} < 0.01,$$

$$\boxed{6}$$

28. Use Taylor's Inequality to determine the interval of x-values such that the third-degree Maclaurin polynomial $T_3(x)$ for $\sin x$ approximates $\sin x$ to within $\dfrac{1}{384}$.

Solution: Since this is a Maclaurin polynomial, $a = 0$. We want to find the largest interval $[-d, d]$ so that for all x in this interval, $|R_3(x)| \leq \frac{1}{384}$. For $f(x) = \sin x$, the fourth derivative is bounded by $M = 1$ on $[-d, d]$, so Taylor's Inequality gives

$$|R_3(x)| \leq \frac{M}{4!}|x|^4 \leq \frac{1}{24}d^4.$$

Now we solve

$$\begin{aligned} \frac{1}{24}d^4 &\leq \frac{1}{384} \\ d^4 &\leq \frac{1}{16} \\ d &\leq \frac{1}{2}. \end{aligned}$$

Therefore, we need $d \leq \frac{1}{2}$, and the interval of x-values is

$$\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right]}.$$