Final Study Guide

MATH2300 - Calculus II

Spring 2025

Contents

Topics Covered

The final exam assesses a broad range of topics from Calculus II. Each topic below includes a brief description of the associated concepts and skills.

Integration Techniques

- **u-substitution:** A foundational method for reversing the chain rule and simplifying integrals by changing variables.
- **Integration by parts:** A technique based on the product rule for differentiation, useful for integrating products of functions.
- **Partial fractions:** A strategy for decomposing rational functions into simpler fractions that can be integrated individually.
- **Trigonometric integrals:** Involves using trigonometric identities to integrate expressions containing sin, cos, tan, etc.
- **Trigonometric substitution:** A method for integrating expressions involving square roots by substituting trigonometric functions.

Applications of Integration

- **Improper integrals:** Integrals with infinite limits or discontinuous integrands, evaluated using limits.
- **Volume of solids:** Methods such as the disk, washer, and shell techniques to compute the volume of solids of revolution or known cross-section.
- **Average value of a function:** Calculates the mean output of a continuous function over an interval.
- **Work problems:** Models the work done by a force (e.g., in stretching a spring or lifting a rope or liquid) using integrals.
- **Center of mass:** Uses integrals to determine the average position (centroid) of a system with uniform or variable density.

Parametric and Polar Equations

- **Parametric curves:** Curves defined by separate equations for $x(t)$ and $y(t)$, including slope, tangent lines, and arc length.
- **Polar curves:** Curves defined by $r = f(\theta)$; includes plotting, finding tangent lines, arc length, and computing enclosed area.

Sequences and Series

- **Understanding sequences and series:** Examines convergence through partial sums and visual behavior of infinite processes.
- **Telescoping series:** A series where terms cancel in a way that makes the sum easy to evaluate.
- **Convergence tests:** Methods to determine whether a series converges or diverges:
  - **Divergence Test:** Uses the limit of the terms to identify divergence.
  - **Geometric Series Test:** Applies to series of the form ar^n.
  - **Integral Test:** Compares a series to an improper integral.
  - **p-series Test:** Tests series of the form $\sum 1/n^p$.

  - **Comparison Tests:** Compares a given series to a known benchmark.
  - **Alternating Series Test:** Applies to series whose terms alternate in sign and decrease in magnitude.
  - **Ratio Test:** Uses the ratio between successive terms to assess convergence.

- **Absolute vs. conditional convergence:** Distinguishes series that converge absolutely (when all terms are made positive) from those that only converge conditionally.

Power Series and Taylor Series

- **Radius and interval of convergence:** Identifies where a power series converges using the Ratio Test.
- **Creating new power series:** Uses algebraic manipulation and substitution to derive new series from known ones.
- **Using known Maclaurin series:** Substitutes or transforms standard series (like those for e^x, $\sin x$, etc.) to match new functions.
- **Taylor polynomials and series:** Represents a function locally as a polynomial or infinite series centered at a point a.

Series Remainder Estimates

- **Alternating Series Remainder Estimate:** Gives an upper bound on error when truncating an alternating series.
- **Remainder bound from the Integral Test:** Uses improper integrals to bound the remainder of positive decreasing series.
- **Taylor's Inequality:** Provides a bound on the error when approximating a function with a Taylor polynomial.

Differential Equations

- **Separable differential equations:** Solves differential equations by separating variables and integrating both sides.
- **Initial value problems:** Uses given conditions to find particular solutions to differential equations.

11.8 Power Series

Definition. A **power series** centered at a is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where c_n are constants called the **coefficients**, a is the **center**, and x is the variable. The series may converge for some values of x and diverge for others.

- **Radius of Convergence R:**
  - The power series converges absolutely when $|x-a| < R$ and diverges when $|x-a| > R$.
  - If $|x-a| = R$, convergence must be checked separately at each endpoint.
  - R can often be found using the **Ratio Test**:

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \quad \text{(if the limit exists).}$$

- **Interval of Convergence:**
  - The interval may be a single point, a finite interval, or the entire real line.
  - Typically has the form $(a-R, a+R)$, $[a-R, a+R]$, $(a-r, a+R]$, or $[a-r, a+R)$, depending on endpoint behavior.

Question Type	**Description**
1. Find Radius and Interval of Convergence	Compute radius R and interval I using the Ratio Test.
2. Determine Radius from Convergence or Divergence Behavior	Infer R based on where the series converges or diverges.
3. Identify Center and Radius from Interval or Series Form	Find center and radius from interval descriptions or series structure.
4. Analyze Specific Points Relative to Convergence	Determine whether a given point lies inside, outside, or on the boundary of convergence.
5. Endpoint and Boundary Behavior	Identify where convergence must be checked separately (endpoints) or is not guaranteed.

Free Response Practice

For each series below, find the radius of convergence R and the interval of convergence I.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n \cdot 3^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x+4)^n}{7^n}$
3. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^{2n}}{5^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3x)^n}{n^2}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$
7. $\displaystyle\sum_{n=2}^{\infty} \frac{(x-7)^n}{n\ln(n)}$
8. $\displaystyle\sum_{n=0}^{\infty} \frac{n!(x+1)^n}{n^n}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^{2n}}{n4^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-5)^n}{2^n(n+1)}$

Multiple Choice Practice

1. The power series
$$\sum_{n=0}^{\infty} a_n(x-1)^n$$
converges at $x = 3$. At which of the following x-values must the given power series also converge?
   (A) $x = 4$
   (B) $x = -2$
   (C) $x = 0$
   (D) $x = 5$
   (E) $x = -5$

2. The power series
$$\sum_{n=0}^{\infty} b_n(x-4)^n$$
converges at $x = 6$. At which of the following x-values must the given power series also converge?
   (A) $x = 2$
   (B) $x = 5$
   (C) $x = 8$
   (D) $x = 10$
   (E) $x = 1$

3. Suppose the power series $\sum a_n(x-1)^n$ converges for $-2 < x < 4$. Which of the following statements is true?
   (A) The radius of convergence is 2.
   (B) The radius of convergence is 3.
   (C) The radius of convergence is 4.
   (D) The center is at $x = 0$.
   (E) The center is at $x = -1$.

4. Which of the following would **definitely** cause a power series centered at $x = 2$ with radius 3 to diverge?
   (A) Evaluating at $x = 4$
   (B) Evaluating at $x = 1$
   (C) Evaluating at $x = 5$
   (D) Evaluating at $x = -2$
   (E) Evaluating at $x = -1$

5. Suppose the radius of convergence of $\sum a_n(x+4)^n$ is 6. Which value must be checked separately to determine convergence?
   (A) $x = 6$
   (B) $x = 4$
   (C) $x = -10$
   (D) $x = 10$
   (E) $x = -2$

6. Suppose $\sum p_n(x+2)^n$ has radius of convergence 4. At which of the following points is convergence **not guaranteed**?
   (A) $x = 0$
   (B) $x = 1$
   (C) $x = -5$
   (D) $x = -1$
   (E) $x = 2$

11.9 Representing Functions as Power Series

Geometric Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

Substitution & Algebraic Manipulation

- Replace $x \mapsto ax$: $\dfrac{1}{1-ax} = \sum a^n x^n$, $|ax| < 1$.
- Factor constants: $\dfrac{1}{b-x} = \dfrac{1}{b}\dfrac{1}{1-\frac{x}{b}} = \sum_{n=0}^{\infty} b^{-n-1} x^n$, $|x| < |b|$.
- Combine series: multiply by x^k, add/subtract termwise.

Term-by-Term Differentiation / Integration If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius R, then on $(a-R, a+R)$

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n\, (x-a)^{n-1}$$

$$\int f(x)\, dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

Radius of convergence remains R (endpoints may change).

Logarithm & Arctangent

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad |x| < 1,$$

integrate $\frac{1}{1+x} = \sum (-1)^n x^n$ and set $C = 0$.

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1,$$

integrate $\frac{1}{1+x^2} = \sum (-1)^n x^{2n}$ and set $C = 0$.

Convergence Facts

- Radius R from Ratio Test: $R = \lim_{n\to\infty} \left| \dfrac{c_n}{c_{n+1}} \right|$.
- Interval is $(a-R, a+R)$; test $x = a \pm R$ separately.
- Within interval, series defines a *continuous*, infinitely-differentiable function.

Question Type	**Description**
1. Geometric Series Substitution	Express a rational function as a geometric series by substitution and/or factoring.
2. Series from Known Maclaurin Series	Modify a known Maclaurin series (like $\ln(1+x)$, $\arctan(x)$) to match the given function.
3. Integrating a Power Series	Represent an integral involving elementary functions by integrating a known series term-by-term.

Free Response Practice

1. $f(x) = \dfrac{x}{1-3x}$
2. $f(x) = \dfrac{4}{7+2x}$
3. $f(x) = \dfrac{1}{9-x^2}$
4. $f(x) = \dfrac{x^2}{1+x}$
5. $f(x) = \dfrac{1}{5-x}$
6. $f(x) = \ln(1+x^2)$
7. $f(x) = \arctan(2x)$
8. $f(x) = \int_0^x \dfrac{1}{1+t^4}\,dt$
9. $f(x) = \int_0^x \dfrac{t}{1+t^2}\,dt$

Multiple Choice Practice

1. Which of the following is the correct power series representation for $\dfrac{1}{1+x}$?

   (A) $\sum_{n=0}^{\infty}(-1)^n x^n$

   (B) $\sum_{n=0}^{\infty} x^n$

   (C) $\sum_{n=0}^{\infty}(-1)^n (x-1)^n$

   (D) $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$

   (E) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^n}{n}$

2. What is the power series representation for $\arctan(x)$?

   (A) $\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$

   (B) $\sum_{n=0}^{\infty} x^{2n+1}$

   (C) $\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$

   (D) $\sum_{n=1}^{\infty}(-1)^n \dfrac{x^{2n+1}}{n}$

   (E) $\sum_{n=1}^{\infty} \dfrac{x^{2n}}{n}$

3. Suppose $\ln(1+x)$ is expanded as a power series centered at 0. What is the radius of convergence?

   (A) 1

   (B) 2

   (C) 1/2

   (D) 3

   (E) Infinite

4. Suppose $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Which series represents $\dfrac{1}{1-2x}$?

   (A) $\sum_{n=0}^{\infty} \left(\dfrac{x}{2}\right)^n$

   (B) $\sum_{n=0}^{\infty} 2^n x^n$

   (C) $\sum_{n=0}^{\infty} 2x^n$

   (D) $\sum_{n=0}^{\infty} 2nx^n$

   (E) $\sum_{n=1}^{\infty} 2^n x^n$

5. What is the power series representation of $\int_0^x \dfrac{t}{1+t^2}\, dt$?

   (A) $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+2}}{2n+2}$

   (B) $\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$

   (C) $\sum_{n=0}^{\infty} \dfrac{x^{2n+2}}{2n+2}$

   (D) $\sum_{n=0}^{\infty}(-1)^n x^{2n}$

   (E) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}x^{2n}}{n}$

6. If $\dfrac{1}{1-x}$ has radius of convergence 1, what is the radius of convergence for $\dfrac{1}{1-5x}$?

   (A) 5

   (B) 1/5

   (C) 2

   (D) 1/2

   (E) 1

11.10 Taylor Series

Taylor Series at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad c_n = \frac{f^{(n)}(a)}{n!}, \quad |x-a| < R.$$

Maclaurin Series (special case $a = 0$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \quad |x| < R.$$

Finding a Series

1. Compute $f^{(n)}(a)$ for $n = 0, 1, 2, \ldots$.
2. Form coefficients $c_n = f^{(n)}(a)/n!$.
3. Write $\sum_{n=0}^{\infty} c_n (x-a)^n$.
4. Determine radius R via Ratio Test:
$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|.$$

Common Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad R = \infty,$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad R = \infty,$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad R = \infty,$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad R = 1,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad R = 1,$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad R = 1.$$

Question Type	**Description**
1. Construct Taylor or Maclaurin Series	Build a Taylor or Maclaurin series using given derivative values or a general pattern.
2. Build Series from Known Maclaurin Series	Start from a known Maclaurin series and modify it by substitution or multiplication to find a new series.
3. Find Specific Coefficients	Find the coefficient of a specific term x^n in a given power series.
4. Integrate Power Series	Integrate a known power series term-by-term to find a new series.
5. Find the Sum of a Series	Recognize the sum of a given series as a known elementary function.
6. Center Shifts and Pattern Adjustments	Adjust a Maclaurin series to a Taylor series centered at $a \neq 0$ and find new intervals of convergence.
7. Manipulate Series by Differentiation or Multiplication	Differentiate or multiply a power series term-by-term to produce a related series.
8. Find Specific Derivatives from Series	Find values like $f^{(n)}(a)$ by using the structure of the Taylor series.

Free Response Practice

1. Suppose f satisfies $f^{(n)}(0) = (-1)^n n!$ for all $n \geq 0$. Find the Maclaurin series for $f(x)$.
2. Starting from the Maclaurin series for $\frac{1}{1-x}$, find a power series for $\frac{1}{1-4x}$ and state its radius of convergence.
3. Use the Maclaurin series for $\ln(1+x)$ to find a power series for $\ln(1-3x)$.
4. Find the coefficient of x^5 in the Maclaurin series for e^{3x}.
5. Find the coefficient of x^{10} in the power series representation of $\arctan(x^2)$.
6. Evaluate $\displaystyle\int \frac{1}{1+x^2}\,dx$ as a power series.
7. Use a power series to evaluate $\displaystyle\int e^{-x^2}\,dx$ (leave as an infinite series).
8. Find a power series representation for $\displaystyle\int \sin(x^2)\,dx$.
9. Evaluate the series
$$\sum_{n=0}^{\infty} \frac{4^n}{n!}.$$
10. Evaluate the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}.$$
11. Evaluate the series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$
12. Find the Taylor series for $f(x) = \frac{1}{1-(x-2)}$ centered at $a = 2$.
13. Find the Taylor series for $f(x) = \ln(1+(x-3))$ centered at $a = 3$.
14. Find a power series representation for xe^x by manipulating the Maclaurin series for e^x.
15. Suppose $f(x)$ has a Maclaurin series $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$. Find $f^{(55)}(0)$.
16. Suppose the coefficient of $(x-2)^7$ in the Taylor series for $f(x)$ centered at $a = 2$ is $\frac{1}{56}$. Find $f^{(7)}(2)$.

Multiple Choice Practice

1. Which of the following correctly describes the beginning of the Maclaurin series for a function $f(x)$ with $f(0) = 1$, $f'(0) = 0$, $f''(0) = 2$, $f^{(3)}(0) = 0$, $f^{(4)}(0) = 4$?

   (A) $f(x) = 1 + x + x^2 + x^3 + \cdots$

   (B) $f(x) = 1 + x^2 + \dfrac{x^4}{2} + \cdots$

   (C) $f(x) = 1 + x^2 + \dfrac{x^3}{6} + \cdots$

   (D) $f(x) = 1 + x^2 + \dfrac{x^4}{6} + \cdots$

   (E) $f(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \cdots$

2. The radius of convergence for $\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$ is:

   (A) 1

   (B) 3

   (C) Infinite

   (D) $\frac{1}{3}$

   (E) 0

3. Which of the following series has radius of convergence $\frac{1}{2}$?

   (A) $\sum x^n$

   (B) $\sum (2x)^n$

   (C) $\sum (x/2)^n$

   (D) $\sum (-x)^n$

   (E) $\sum (3x)^n$

4. Which of the following represents $\displaystyle\int \frac{1}{1+x^2}\,dx$ as a power series?

   (A) $\displaystyle\sum_{n=0}^{\infty} x^{2n+1} + C$

   (B) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2n} + C$

   (C) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n} + C$

   (D) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n+1} + C$

   (E) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$

5. Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

   (A) e^{2x}

   (B) e^{x^2}

   (C) $\frac{1}{1-2x}$

   (D) $\sin(2x)$

   (E) $\cos(2x)$

6. Suppose $\sum_{n=0}^{\infty} c_n (x-3)^n$ converges when $|x-3| < 2$. Which interval describes where the series converges?

   (A) $(-2, 2)$

   (B) $(1, 5)$

   (C) $(3, 5)$

   (D) $(1, 3)$

   (E) $(2, 4)$

7. Suppose the Maclaurin series for $f(x)$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   What is $f^{(4)}(0)$?

   (A) 1

   (B) 4

   (C) 6

   (D) 24

   (E) 0

8. Which of the following series represents $\cos(x)$?

   (A) $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$

   (B) $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$

   (C) $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

   (D) $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^n}{n}$

   (E) $\sum_{n=0}^{\infty} x^{2n}$

11.11 Taylor Remainder Estimate

Theorem (Taylor's Inequality)**.** If $\left|f^{(n+1)}(x)\right| \leq M$ for all $|x-a| \leq d$, then

$$\left|R_n(x)\right| < \frac{M}{(n+1)!}\left|x-a\right|^{n+1}.$$

Use this to choose n so that the error is below a desired tolerance.

- To prove f equals its Taylor series, show $\lim_{n\to\infty} R_n(x) = 0$ via either Taylor's Inequality or the alternative remainder forms.
- For alternating-series remainders, the first omitted term often bounds $\left|R_n(x)\right|$.
- Always check the interval $|x-a| < R$, where R is the radius of convergence of the series.

Question Type	**Description**
1. Find n for Target Error	Determine how large n must be so that the remainder $\|R_n(x)\|$ is below a specified bound.
2. Find Taylor Polynomial $T_n(x)$	Construct the Taylor polynomial of given degree n centered at a given point a.
3. Bound $\|R_n(x)\|$ Over an Interval	Use Taylor's Inequality to bound the error $\|R_n(x)\|$ over a specified interval.
4. Alternating Series Estimation	Apply the Alternating Series Estimation Theorem to bound the error for alternating series.
5. Find Interval for Specified Accuracy	Determine the set of x-values where the Taylor polynomial approximates the function within a given error.
6. Determine When Taylor Series Matches Function	Recognize when the Taylor series converges to $f(x)$ and when it does not.
7. Graphical Confirmation of Error Bound	Use a graph to confirm that the actual error is within the theoretical bound.
8. Find Number of Terms Needed for Precision	Find how many terms (starting from $n = 0$) are needed for a Taylor series approximation to achieve a specified precision.

Free Response Practice

1. Find the minimum degree n necessary so that the Maclaurin polynomial for $\sin(x)$ approximates $\sin(0.2)$ with an error less than 10^{-6}.

2. Find the minimum degree n necessary so that the Taylor polynomial for e^x centered at $a = 0$ approximates $e^{0.5}$ within an error less than 0.0001.

3. Find the Taylor polynomial of degree 3 centered at $a = 0$ for $f(x) = \cos(x)$.

4. Find an upper bound for the error $|R_2(x)|$ when approximating e^x on the interval $0 \leq x \leq 0.5$ using the degree 2 Taylor polynomial centered at $a = 0$.

5. Find an upper bound for $|R_3(x)|$ when approximating $\sin(x)$ on $-0.2 \leq x \leq 0.2$ with a Taylor polynomial centered at $a = 0$.

6. Use the Alternating Series Estimation Theorem to estimate the error when approximating $\arctan(0.5)$ by the sum of the first two nonzero terms of its Maclaurin series.

7. Determine for which values of x the degree 2 Taylor polynomial for $\ln(1+x)$ approximates $\ln(1+x)$ within 0.0005 error.

Multiple Choice Practice

1. What is the smallest degree n needed so that the Maclaurin polynomial for e^x approximates $e^{0.2}$ within error less than 0.001?

   (A) $n = 2$
   (B) $n = 3$
   (C) $n = 4$
   (D) $n = 5$
   (E) $n = 6$

2. Find the Taylor polynomial of degree 2 centered at $a = 0$ for $\cos(x)$.

   (A) $1 - x + \frac{x^2}{2}$
   (B) $1 - \frac{x^2}{2}$
   (C) $1 + x^2$
   (D) $1 - \frac{x^2}{3}$
   (E) $1 - x^2$

3. Which expression best estimates the maximum error when approximating e^x by its degree 2 Taylor polynomial centered at $a = 0$ for $x = 0.1$?

   (A) $\dfrac{e^{0.1}0.1^2}{2!}$
   (B) $\dfrac{e^{0.1}0.1^2}{3!}$
   (C) $\dfrac{e^{0.1}0.1^3}{3!}$
   (D) $\dfrac{e^{0.1}0.1^4}{4!}$
   (E) $\dfrac{e^{0.1}0.1^3}{2!}$

4. Which expression estimates the maximum error when approximating $\sin(x)$ with a degree 3 Taylor polynomial centered at 0 on $-0.1 \le x \le 0.1$?

   (A) $\dfrac{(0.1)^2}{2!}$
   (B) $\dfrac{(0.1)^3}{3!}$
   (C) $\dfrac{(0.1)^4}{4!}$
   (D) $\dfrac{(0.1)^5}{5!}$
   (E) $\dfrac{(0.1)^6}{6!}$

5. Using the Alternating Series Estimation Theorem, which bound best estimates the error for approximating $\arctan(0.5)$ with the first three nonzero terms?

   (A) Less than $(0.5)^3$
   (B) Less than $\dfrac{(0.5)^4}{4}$
   (C) Less than $\dfrac{(0.5)^5}{5}$
   (D) Less than $\dfrac{(0.5)^6}{6}$
   (E) Less than $\dfrac{(0.5)^7}{7}$

6. Which value provides an upper bound for the error when approximating $\ln(1+x)$ at $x = 0.2$ using the first two nonzero terms?

   (A) $(0.2)^2$
   (B) $(0.2)^3$
   (C) $\dfrac{(0.2)^2}{2}$
   (D) $\dfrac{(0.2)^3}{3}$
   (E) $\dfrac{(0.2)^2}{3}$

7. Find the largest interval of those listed below where the degree 2 Taylor polynomial for e^x centered at 0 approximates e^x within an error less than 0.001?

   (A) $|x| < 0.05$
   (B) $|x| < 0.1$
   (C) $|x| < 0.5$
   (D) $|x| < 1$
   (E) $|x| < 2$

8. Find the largest interval of those listed below where the degree 3 Taylor polynomial for $\sin(x)$ approximates $\sin(x)$ within 10^{-4} error.

   (A) $|x| < 0.1$
   (B) $|x| < 0.5$
   (C) $|x| < 1$
   (D) $|x| < 2$
   (E) $|x| < 3$

9.1 Differential Equations

Definition. A **differential equation** is an equation involving an unknown function and one or more of its derivatives.

- **Order**: The highest derivative present.
- **General Solution**: The full family of solutions.
- **Particular Solution**: A specific solution satisfying an initial condition.
- **Initial-Value Problem (IVP)**: A differential equation plus an initial condition like $y(t_0) = y_0$.

Remark. To verify that $y = f(x)$ is a solution of a differential equation:

1. Differentiate $f(x)$ as needed.
2. Substitute into the differential equation.
3. Simplify to check if both sides are equal.

Question Type	Description
1. Verify a Given Function is a Solution	Substitute y, y', or y'' into the differential equation and check if the function satisfies it.
2. Determine Which Functions are Solutions (Select All That Apply)	Analyze multiple given functions and determine which ones solve the differential equation.
3. Solve a Differential Equation for a General Family of Solutions	Integrate directly to find the general solution, including a constant C.
4. Solve for a Particular Solution Given an Initial Condition	Solve a differential equation and use an initial condition to find the specific value of C.
5. Analyze Graphs to Determine If a Graph Could Represent a Solution	Use properties like slope, monotonicity, and concavity implied by y' or y'' to assess whether a graph could be a solution.

Free Response Practice

1. Verify that $y = e^{3x}$ is a solution to the differential equation:
$$y' = 3y$$
2. Verify that $y = \sin(2x)$ is a solution to the differential equation:
$$y'' + 4y = 0$$
3. Verify that $y = x^2 + 1$ satisfies the differential equation:
$$y' = 2x$$
4. Solve the differential equation $y' = 2x$ given that $y(1) = 5$.
5. Find the particular solution to $\frac{dy}{dx} = 3e^x$ satisfying $y(0) = 2$.
6. Solve $\frac{dy}{dx} = \sin(x)$ with the initial condition $y\left(\frac{\pi}{2}\right) = 0$.

Multiple Choice Practice

1. Verify whether $y = e^{2x}$ is a solution to $y' = 2y$.
   (A) Yes, it satisfies the differential equation.
   (B) No, it does not satisfy the differential equation.

2. Is $y = x^2 + 1$ a solution to the differential equation $y' = 2x$?
   (A) Yes
   (B) No

3. Which of the following functions satisfy $y' = 3y$? (Select all that apply.)
   (a) $y = e^{3x}$
   (b) $y = 2e^{3x}$
   (c) $y = e^{-3x}$
   (d) $y = 3e^{x}$

4. Which of the following satisfy $y'' + y = 0$? (Select all that apply).
   (a) $y = \sin(x)$
   (b) $y = \cos(x)$
   (c) $y = e^{x}$
   (d) $y = \sin(x) + \cos(x)$

5. Solve the differential equation $y' = 6x$.
   (A) $y = 6x + C$
   (B) $y = 3x + C$
   (C) $y = 12x + C$
   (D) $y = 3x^2 + C$
   (E) $y = 6x^2 + C$

6. Solve the differential equation $\frac{dy}{dx} = \cos(x)$.
   (A) $y = \cos(x) + C$
   (B) $y = \sin(x) + C$
   (C) $y = -\sin(x) + C$
   (D) $y = -\cos(x) + C$
   (E) $y = x\cos(x) + C$

7. Solve the differential equation $y' = 2x$ given that $y(1) = 6$.
   (A) $y = x^2 + 2$
   (B) $y = x^2 + 3$
   (C) $y = x^2 + 4$
   (D) $y = x^2 + 5$
   (E) $y = 2x^2 + 5$

8. Find the particular solution to $\frac{dy}{dx} = e^{x}$ satisfying $y(0) = 3$.
   (A) $y = e^{x} + 3$
   (B) $y = e^{x} + 2$
   (C) $y = e^{x} + 1$
   (D) $y = e^{x} - 3$
   (E) $y = 3e^{x}$

9. True or False: If $y' = 2y$, then any solution graph must be increasing wherever $y > 0$.

10. True or False: If $y' = -y$, then the solution curves are always decreasing when $y > 0$.

9.3 Separable Differential Equations

Definition. A differential equation is **separable** if it can be written in the form

$$\frac{dy}{dx} = g(x)h(y)$$

This allows separation of variables:

$$\frac{1}{h(y)}\,dy = g(x)\,dx$$

Steps for Solving a Separable Differential Equation

1. Rewrite in the form $\frac{dy}{dx} = g(x)h(y)$.
2. Separate variables: $\frac{1}{h(y)}\,dy = g(x)\,dx$.
3. Integrate both sides.
4. Solve explicitly for y, if possible.
5. Apply any initial condition to determine the constant C.

Remark. If the integral involves $\int \frac{1}{y}\,dy$, the solution includes a logarithm:

$$\int \frac{1}{y}\,dy = \ln|y| + C$$

Be careful with absolute values when solving for y.

Question Type	**Description**
1. Identify Whether an Equation Is Separable	Determine whether a differential equation can be separated into the form $h(y)\,dy = g(x)\,dx$.
2. Solve Separable Differential Equations (General Solutions)	Solve a separable differential equation by separating variables and finding the general solution (including a constant C).
3. Solve Separable Differential Equations (Particular Solutions)	Solve a separable differential equation using an initial condition to determine the specific value of C.
4. Solve Separable Differential Equations (Implicit Solutions)	Solve a separable differential equation where the solution may remain implicit if solving explicitly for y is difficult or unnecessary.

Free Response Practice

1. Solve the differential equation $y' = xy$ by finding the general solution.
2. Solve the differential equation $y' = \dfrac{2y}{x}$ by finding the general solution.
3. Solve the differential equation $y' = x(1 + y^2)$ by finding the general solution.
4. Solve the differential equation: $y' = y^2$ with the initial condition $y(0) = 1$.
5. Solve the differential equation: $y' = (1 - x^2)y$ with $y(0) = 2$.
6. Solve the differential equation: $y' = x^2(1 + y)$ given that $y(1) = 0$.
7. Solve the differential equation: $y' = \dfrac{y}{x}$ and leave your answer in implicit form if necessary.
8. Solve the differential equation: $y' = \dfrac{x}{y}$ and leave your answer in implicit form if necessary.
9. Solve the differential equation: $y' = \dfrac{2y}{x+1}$ and leave your answer in implicit form if necessary.

Multiple Choice Practice

1. Which of the following differential equations is separable?

   (A) $\frac{dy}{dx} = x^2 + y^2$
   (B) $\frac{dy}{dx} = e^x + y$
   (C) $\frac{dy}{dx} = xy$
   (D) $\frac{dy}{dx} = \tan(x+y)$
   (E) $\frac{dy}{dx} = \ln(x+y)$

2. Which of the following is NOT separable?

   (A) $\frac{dy}{dx} = \frac{x}{1+y^2}$
   (B) $\frac{dy}{dx} = x(1+y^2)$
   (C) $\frac{dy}{dx} = x+y$
   (D) $\frac{dy}{dx} = \frac{y}{x}$
   (E) $\frac{dy}{dx} = \frac{y^2}{x}$

3. Solve for the general solution to $\frac{dy}{dx} = x^2 y$.

   (A) $y = Ce^{x^2}$
   (B) $y = Cx^2$
   (C) $y = Ce^{x^3/3}$
   (D) $y = Cxe^x$
   (E) $y = Cx^3 + 1$

4. Solve for the general solution to $\frac{dy}{dx} = \frac{3y}{x}$.

   (A) $y = Cx^3$
   (B) $y = Cx$
   (C) $y = Cx^2$
   (D) $y = Ce^{3x}$
   (E) $y = Cx^{-3}$

5. Find the particular solution to $\frac{dy}{dx} = y^2$ with initial condition $y(0) = 2$.

   (A) $y = \frac{1}{2-x}$
   (B) $y = \frac{1}{x+2}$
   (C) $y = 2 + x$
   (D) $y = 2 - x$
   (E) $y = \frac{1}{x-2}$

6. Find the particular solution to $\frac{dy}{dx} = (1-x^2)y$ with $y(0) = 3$.

   (A) $y = 3e^{-x+x^3/3}$
   (B) $y = 3e^{x-x^3/3}$
   (C) $y = 3e^{x^2/2}$
   (D) $y = 3e^{x-x^2}$
   (E) $y = 3e^{x^3-x}$

7. Solve the separable differential equation $\frac{dy}{dx} = \frac{2y}{x}$.

   (A) $y = Cx$
   (B) $y = Cx^2$
   (C) $y = Cx^3$
   (D) $y = Cx^4$
   (E) $y = C\ln(x)$

8. Solve the separable differential equation $\frac{dy}{dx} = \frac{x^2}{y}$.

   (A) $y^2 = \frac{2x^3}{3} + C$
   (B) $y = Cx^2$
   (C) $y = \frac{x^3}{3}$
   (D) $y^2 = x^2 + C$
   (E) $y = e^{x^2}$

Solutions

11.8 Power Series (Solutions)

Free Response Practice

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n \cdot 3^n}$

   Apply the Ratio Test:

   $$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(2x-5)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n \cdot 3^n}{(2x-5)^n} \right| \\ &= \left| \frac{2x-5}{3} \right| \end{aligned}$$

   So the radius is $R = \frac{3}{2}$, and the center is $x = \frac{5}{2}$. Solving $|2x-5| < 3$ gives the open interval $(1, 4)$. At $x = 1$, we get $\sum \frac{(-1)^n}{n}$ — converges. At $x = 4$, we get $\sum \frac{1}{n}$ — diverges.

   $$\boxed{R = \frac{3}{2}, \quad I = [1, 4)}$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x+4)^n}{7^n}$

   Apply the Ratio Test:

   $$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{x+4}{7} \right| \\ &= \left| \frac{x+4}{7} \right| \end{aligned}$$

   So $R = 7$, centered at $x = -4$, giving interval $(-11, 3)$. At both endpoints, the series converges by alternating series or comparison.

   $$\boxed{R = 7, \quad I = [-11, 3]}$$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$

   This is the Maclaurin series for e^{x-1}, which converges for all real x.

   $$\boxed{R = \infty, \quad I = (-\infty, \infty)}$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^{2n}}{5^n}$

   This is geometric with ratio $r = \frac{(x+2)^2}{5}$. Require $|r| < 1 \Rightarrow (x+2)^2 < 5$, so:

   $$\begin{gathered} -\sqrt{5} < x + 2 < \sqrt{5} \\ -2 - \sqrt{5} < x < -2 + \sqrt{5} \end{gathered}$$

   $$\boxed{R = \sqrt{5}, \quad I = (-2 - \sqrt{5}, -2 + \sqrt{5})}$$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3x)^n}{n^2}$

   Apply the Ratio Test:

   $$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x)^n} \right| \\ &= 3|x| \end{aligned}$$

   So $R = \frac{1}{3}$, centered at $x = 0$. At $x = \pm\frac{1}{3}$, alternating series converge.

   $$\boxed{R = \frac{1}{3}, \quad I = [-\tfrac{1}{3}, \tfrac{1}{3}]}$$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$

   Apply the Ratio Test:

   $$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{x}{n+3} \right| = 0 \end{aligned}$$

   So the series converges for all x.

   $$\boxed{R = \infty, \quad I = (-\infty, \infty)}$$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{(x-7)^n}{n\ln(n)}$

Apply the Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{(x-7)^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n\ln(n)}{(x-7)^n} \right| \\ = |x-7|$$

So $R = 1$, centered at $x = 7$. Endpoints converge by comparison and alternating series tests.

$$\boxed{R = 1, \quad I = [6, 8]}$$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{n!\,(x+1)^n}{n^n}$

Let

$$a_n = \frac{n!\,(x+1)^n}{n^n}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = |x+1|\,\frac{(n+1)\,n^n}{(n+1)^{n+1}} = |x+1|\,\left(\tfrac{n}{n+1}\right)^n,$$

and $\lim_{n\to\infty}(n/(n+1))^n = 1/e$. Hence

$$L = \frac{|x+1|}{e} < 1 \quad \Longrightarrow \quad R = e, \quad \text{center } x = -1.$$

Check the endpoint $x = -1 + e$. Then $x + 1 = e$ and

$$a_n = \frac{n!\,e^n}{n^n}.$$

Since for all $n \geq 1$,

$$\frac{a_{n+1}}{a_n} = e\left(\tfrac{n}{n+1}\right)^n > e \cdot \frac{1}{e} = 1,$$

the sequence $\{a_n\}$ is strictly increasing and $a_1 = e > 0$. Thus $\lim_{n\to\infty} a_n \geq e \neq 0$, so the series diverges by the Divergence Test.

Check the endpoint $x = -1 - e$. Now $x + 1 = -e$ and $|a_{n+1}/a_n| > 1$ still, so $\lim |a_n| \neq 0$, and the series again diverges.

$$\boxed{R = e, \quad I = (-1-e, -1+e).}$$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^{2n}}{n \cdot 4^n}$

Let

$$a_n = \frac{(2x-3)^{2n}}{n\,4^n}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x-3|^2\,(n/(n+1))}{4} \xrightarrow[n\to\infty]{} \frac{(2x-3)^2}{4}.$$

Convergence requires

$$\frac{(2x-3)^2}{4} < 1 \quad \Longrightarrow \quad |2x-3| < 2 \quad \Longrightarrow \quad \tfrac{1}{2} < x < \tfrac{5}{2}.$$

Hence the radius about the center $x = \frac{3}{2}$ is

$$\boxed{R = 1, \quad I = \left(\tfrac{1}{2}, \tfrac{5}{2}\right).}$$

At $x = \frac{5}{2}$, $2x - 3 = 2$, so

$$a_n = \frac{2^{2n}}{n\,4^n} = \frac{4^n}{n\,4^n} = \frac{1}{n},$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
At $x = \frac{1}{2}$, $2x - 3 = -2$, so

$$a_n = \frac{(-2)^{2n}}{n\,4^n} = \frac{4^n}{n\,4^n} = \frac{1}{n},$$

which again diverges.

$$\boxed{I = \left(\tfrac{1}{2}, \tfrac{5}{2}\right).}$$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-5)^n}{2^n(n+1)}$

Let

$$a_n = \frac{n\,(x-5)^n}{2^n\,(n+1)}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x-5)^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{n\,(x-5)^n}\right| \\ = \left|\frac{x-5}{2}\right| \frac{(n+1)^2}{n(n+2)} \xrightarrow[n\to\infty]{} \left|\frac{x-5}{2}\right|.$$

Convergence requires $\left|x-5\right|/2 < 1$, i.e. $3 < x < 7$. Hence

$$\boxed{R = 2, \quad \text{centered at } x = 5.}$$

At $x = 7$: $a_n = \frac{n \cdot 2^n}{2^n(n+1)} = \frac{n}{n+1} \to 1 \neq 0$, so the series diverges.

At $x = 3$: $a_n = \frac{n\,(-2)^n}{2^n(n+1)} = (-1)^n \frac{n}{n+1}$, whose terms do not tend to zero, so the series diverges.

$$\boxed{I = (3, 7).}$$

Multiple Choice Practice

1. **(C)** $x = 0$. Since the power series converges at $x = 3$, its radius satisfies $R \geq 2$. Only $x = 0$ lies in $(-1, 3)$.

2. **(B)** $x = 5$. Convergence at $x = 6$ gives $R \geq |6 - 4| = 2$, so the series converges for $|x - 4| < 2$. At $x = 5$, $|5 - 4| = 1 < 2$.

3. **(B)** Radius $= 3$. The interval of convergence $(-2, 4)$ has center $\frac{-2+4}{2} = 1$ and half-width 3.

4. **(D)** $x = -2$. A series centered at 2 with $R = 3$ converges for $|x - 2| < 3$. At $x = -2$, $|-2-2| = 4 > 3$, so it must diverge.

5. **(C)** $x = -10$. Radius 6 about $x = -4$ covers $|x + 4| < 6$. At $x = -10$, $|-10+4| = 6$ is an endpoint and must be checked; other listed points lie outside or trivially inside.

6. **(E)** $x = 2$. Radius 4 about $x = -2$ covers $|x + 2| < 4$. At $x = 2$, $|2 + 2| = 4$ is an endpoint—convergence there is not guaranteed.

11.9 Representing Functions as Power Series (Solutions)

Free Response Practice

1. $f(x) = \dfrac{x}{1-3x}$

$$\begin{aligned} f(x) &= x \cdot \frac{1}{1-3x} \\ &= x \sum_{n=0}^{\infty} (3x)^n \\ &= \sum_{n=0}^{\infty} 3^n \, x^{n+1}. \end{aligned}$$

Converges when $|3x| < 1$, i.e. $|x| < \dfrac{1}{3}$. $\boxed{R = \frac{1}{3}.}$

2. $f(x) = \dfrac{4}{7+2x}$

$$\begin{aligned} f(x) &= \frac{4}{7} \frac{1}{1+\frac{2x}{7}} \\ &= \frac{4}{7} \sum_{n=0}^{\infty} (-1)^n \left(\tfrac{2x}{7}\right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \, \frac{4 \cdot 2^n}{7^{\,n+1}} \, x^n. \end{aligned}$$

Converges when $\left|\frac{2x}{7}\right| < 1$, i.e. $|x| < \frac{7}{2}$. $\boxed{R = \frac{7}{2}.}$

3. $f(x) = \dfrac{1}{9-x^2}$

$$\begin{aligned} f(x) &= \frac{1}{9} \frac{1}{1-\frac{x^2}{9}} \\ &= \frac{1}{9} \sum_{n=0}^{\infty} \left(\tfrac{x^2}{9}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{9^{\,n+1}}. \end{aligned}$$

Converges when $\left|\frac{x^2}{9}\right| < 1$, i.e. $|x| < 3$. $\boxed{R = 3.}$

4. $f(x) = \dfrac{x^2}{1+x}$

$$\begin{aligned} f(x) &= x^2 \, \frac{1}{1+x} \\ &= x^2 \sum_{n=0}^{\infty} (-1)^n x^n \\ &= \sum_{n=0}^{\infty} (-1)^n \, x^{\,n+2}. \end{aligned}$$

Converges when $|x| < 1$. $\boxed{R = 1.}$

5. $f(x) = \dfrac{1}{5-x}$

$$\begin{aligned} f(x) &= \frac{1}{5} \frac{1}{1-\frac{x}{5}} \\ &= \frac{1}{5} \sum_{n=0}^{\infty} \left(\tfrac{x}{5}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{5^{\,n+1}}. \end{aligned}$$

Converges when $\left|\frac{x}{5}\right| < 1$, i.e. $|x| < 5$. $\boxed{R = 5.}$

6. $f(x) = \ln(1+x^2)$

$$\begin{aligned} \ln(1+x^2) &= \int_0^{x^2} \frac{1}{1+t} \, dt \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \, \frac{(x^2)^n}{n} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \, \frac{x^{2n}}{n}. \end{aligned}$$

Converges when $|x^2| < 1$, i.e. $|x| < 1$. $\boxed{R = 1.}$

7. $f(x) = \arctan(2x)$

$$\begin{aligned} \arctan(2x) &= \int_0^{2x} \frac{1}{1+u^2} \, du \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \, \frac{2^{2n+1}}{2n+1} \, x^{2n+1}. \end{aligned}$$

Converges when $|2x| < 1$, i.e. $|x| < \frac{1}{2}$. $\boxed{R = \frac{1}{2}.}$

8. $f(x) = \displaystyle\int_0^x \frac{1}{1+t^4} \, dt$

$$\begin{aligned} \frac{1}{1+t^4} &= \sum_{n=0}^{\infty} (-1)^n \, t^{4n}, \quad |t| < 1, \\ \int_0^x \frac{1}{1+t^4} \, dt &= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{4n} \, dt \\ &= \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{4n+1}}{4n+1}. \end{aligned}$$

Converges when $|x| < 1$. $\boxed{R = 1.}$

9. $f(x) = \int_0^x \frac{t}{1+t^2}\,dt$

$$\frac{t}{1+t^2} = t\sum_{n=0}^{\infty}(-1)^n t^{2n} = \sum_{n=0}^{\infty}(-1)^n t^{2n+1}, \quad |t| < 1,$$

$$\begin{aligned}\int_0^x \frac{t}{1+t^2}\,dt &= \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n+1}\,dt \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{2n+2}.\end{aligned}$$

Converges when $|x| < 1$. $\boxed{R = 1.}$

Multiple Choice Practice

1. **(A)** $\dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$.

2. **(C)** $\arctan x = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$.

3. **(A)** $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$ has radius 1.

4. **(B)** $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ for $|2x| < 1$.

5. **(A)** Use the geometric series for $\frac{1}{1+t^2}$:

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n\, t^{2n}, \qquad |t| < 1.$$

   Multiply by t and integrate term-by-term:

$$\frac{t}{1+t^2} = \sum_{n=0}^{\infty}(-1)^n\, t^{2n+1},$$
$$\int_0^x \frac{t}{1+t^2}\, dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n+1}\, dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{2n+2}.$$

6. **(B)** If $\frac{1}{1-x}$ has $R = 1$, then $\frac{1}{1-5x}$ has $R = \frac{1}{5}$.

11.10 Taylor Series (Solutions)

Free Response Practice

1. Suppose $f^{(n)}(0) = (-1)^n\, n!$. Then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n \\ &= \sum_{n=0}^{\infty} (-1)^n\, x^n \\ &= \frac{1}{1+x} \end{aligned}$$

Converges for $|x| < 1$.

2. Starting from $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, replace $x \mapsto 4x$:

$$\begin{aligned} \frac{1}{1-4x} &= \sum_{n=0}^{\infty} (4x)^n \\ &= \sum_{n=0}^{\infty} 4^n\, x^n \end{aligned}$$

Radius $R = \frac{1}{4}$.

3. From $\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{u^n}{n}$, set $u = -3x$:

$$\begin{aligned} \ln(1-3x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-3x)^n}{n} \\ &= -\sum_{n=1}^{\infty} \frac{3^n}{n}\, x^n \end{aligned}$$

Valid for $|x| < \frac{1}{3}$.

4. Coefficient of x^5 in e^{3x}:

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

Hence $[x^5] = \frac{3^5}{5!}$.

5. Coefficient of x^{10} in $\arctan(x^2)$:

$$\begin{aligned} \arctan(x^2) &= \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{2k+1} \end{aligned}$$

Solve $4k+2 = 10 \implies k = 2$, so $[x^{10}] = \frac{1}{5}$.

6. $\displaystyle\int \frac{1}{1+x^2}\, dx$ as a series:

$$\begin{aligned} \frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \\ \int \frac{1}{1+x^2}\, dx &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \end{aligned}$$

7. $\displaystyle\int e^{-x^2}\, dx$ as a series:

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \\ \int e^{-x^2}\, dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} + C \end{aligned}$$

8. Find a power-series for $\displaystyle\int \sin(x^2)\, dx$:

$$\begin{aligned} \sin(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}, \\ \int \sin(x^2)\, dx &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!\,(4n+3)} + C. \end{aligned}$$

9. Evaluate the series $\sum_{n=0}^{\infty} \frac{4^n}{n!}$.

$$\sum_{n=0}^{\infty} \frac{4^n}{n!} = e^4$$

10. Evaluate the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = \sin(1)$$

11. Evaluate the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} &= \arctan(1) \\ &= \frac{\pi}{4} \end{aligned}$$

12. Taylor series for $f(x) = \dfrac{1}{1-(x-2)}$ centered at $a = 2$:

$$\frac{1}{1-(x-2)} = \sum_{n=0}^{\infty} (x-2)^n, \quad |x-2| < 1.$$

13. Taylor series for $f(x) = \ln\bigl(1 + (x-3)\bigr)$ at $a = 3$:

$$\ln\bigl(1+(x-3)\bigr) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-3)^n}{n}, \quad |x-3| < 1.$$

14. Power-series for xe^x:

$$\begin{aligned} xe^x &= x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}. \end{aligned}$$

15. If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{3^n}$, then

$$\begin{aligned} \frac{f^{(n)}(0)}{n!} &= \frac{1}{3^n}, \\ f^{(55)}(0) &= \frac{55!}{3^{55}}. \end{aligned}$$

16. If the $(x-2)^7$-coefficient in the Taylor series of f at $a = 2$ is $\frac{1}{56}$, then

$$\begin{aligned} \frac{f^{(7)}(2)}{7!} &= \frac{1}{56}, \\ f^{(7)}(2) &= \frac{7!}{56} = 90. \end{aligned}$$

Multiple Choice Practice

1. **(D)**. The Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n.$$

Given

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = 2, \quad f^{(3)}(0) = 0, \quad f^{(4)}(0) = 4,$$

the nonzero coefficients are

$$\begin{aligned} a_0 &= \frac{f(0)}{0!} = 1, \\ a_2 &= \frac{f''(0)}{2!} = \frac{2}{2} = 1, \\ a_4 &= \frac{f^{(4)}(0)}{4!} = \frac{4}{24} = \frac{1}{6}. \end{aligned}$$

Thus

$$f(x) = 1 + x^2 + \frac{x^4}{6} + \cdots,$$

2. **(C).**

$$\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = e^{3x},$$

which converges for all x.

3. **(B).**

$$\sum (2x)^n$$

has radius $\frac{1}{2}$.

4. **(E).**

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n\, x^{2n}, \quad |x| < 1,$$

$$\int \frac{1}{1+x^2}\, dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.$$

5. **(A).**

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}.$$

6. **(B).** Center 3, radius 2 $\rightarrow$ (1,5).

7. **(A).**

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies f^{(4)}(0) = 1.$$

8. **(A).**

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

11.11 Taylor Remainder Estimate (Solutions)

Free Response Practice

1. Find the minimum degree n so that the Maclaurin polynomial for $\sin x$ approximates $\sin(0.2)$ with error $< 10^{-6}$.

   By Taylor's Inequality,

$$\left|R_n(0.2)\right| \le \frac{M}{(n+1)!}\,|0.2|^{n+1},$$

   where $M = \max_{|t|\le 0.2} |f^{(n+1)}(t)| = 1$. Hence

$$\frac{0.2^{\,n+1}}{(n+1)!} < 10^{-6}.$$

   Test successive n:

$$\begin{aligned} n = 4: \frac{0.2^5}{5!} &= \frac{0.00032}{120} \approx 2.67\times 10^{-6} > 10^{-6} \\ n = 5: \frac{0.2^6}{6!} &= \frac{0.000064}{720} \approx 8.89\times 10^{-8} < 10^{-6}. \end{aligned}$$

   Therefore the smallest suitable degree is

$$\boxed{n = 5.}$$

2. Minimum n so $\left|e^{0.5} - T_n(0.5)\right| < 10^{-4}$.
   Remainder bound:

$$|R_n| \le \frac{e^{0.5}\,0.5^{\,n+1}}{(n+1)!} < 10^{-4}.$$

   Test $n+1 = 3, 4, 5, 6$:

$$\frac{e^{0.5}\,0.5^6}{6!} \approx 3.58\times 10^{-5} < 10^{-4}$$

   so $n+1 = 6 \implies n = 5$.

3. T_3 for $\cos x$ at 0:

$$T_3(x) = \sum_{k=0}^{3} \frac{\cos^{(k)}(0)}{k!}x^k = 1 - \frac{x^2}{2}.$$

4. Error bound for e^x, degree 2 on $[0, 0.5]$:

$$|R_2| \le \frac{e^{0.5}\,(0.5)^3}{3!} = \frac{e^{0.5}\cdot 0.125}{6} \approx 0.0343.$$

5. Error bound for $\sin x$, degree 3 on $[-0.2, 0.2]$:

$$\begin{aligned} |R_3| &\le \frac{1\cdot 0.2^4}{24} \\ &= \frac{0.0016}{24} \approx 6.67\times 10^{-5}. \end{aligned}$$

6. Approximate $\arctan(0.5)$ by $x - \frac{x^3}{3}$.
   Next term bound:

$$|R| \le \frac{|x|^5}{5} = \frac{0.5^5}{5} = 0.00625.$$

7. Determine for which x the degree-2 Taylor polynomial

$$T_2(x) = x - \tfrac{x^2}{2}$$

   for $\ln(1+x)$ approximates $\ln(1+x)$ with error < 0.0005.

   Here $f'''(t) = -2/(1+t)^3$, so on $|t| \le |x| < 1$ we have

$$|f^{(3)}(t)| \le \frac{2}{(1-|x|)^3}.$$

   By Taylor's Inequality,

$$|R_2(x)| \le \frac{M}{3!}\,|x|^3 \le \frac{2}{6\,(1-|x|)^3}\,|x|^3 = \frac{|x|^3}{3\,(1-|x|)^3}.$$

   We require

$$\frac{|x|^3}{3\,(1-|x|)^3} < 0.0005.$$

   Taking cube roots gives

$$\frac{|x|}{1-|x|} < \sqrt[3]{0.0015} \approx 0.1153,$$

   so

$$|x| < 0.1153\,(1-|x|) \quad\implies\quad |x| \le 0.103.$$

   Thus

$$\boxed{|x| < 0.103.}$$

Multiple Choice Practice

1. **(B)** Error bound for $e^{0.2}$ using degree-n Maclaurin:
$$|R_n| \le \frac{e^{0.2}\,(0.2)^{n+1}}{(n+1)!} < 0.001.$$
Test $n+1$:
$$n+1 = 3 : \frac{e^{0.2}0.2^3}{3!} \approx 0.009 > 0.001$$
$$n+1 = 4 : \frac{e^{0.2}0.2^4}{4!} \approx 0.000081 < 0.001.$$
Hence the smallest n is 3.

2. **(B)**. Degree-2 Taylor for $\cos x$ at 0:
$$T_2(x) = 1 - \frac{x^2}{2}.$$

3. **(C)**. For e^x about $a = 0$, Taylor's inequality gives
$$|R_2(0.1)| \le \frac{M}{3!}\,|0.1|^3,$$
where $M = \max_{0\le t\le 0.1} e^t = e^{0.1}$. Thus
$$|R_2(0.1)| \le \frac{e^{0.1}\,(0.1)^3}{6} \approx \frac{(0.1)^3}{3!}.$$

4. **(C)**. Error bound for $\sin x$ degree-3 on $[-0.1, 0.1]$:
$$|R_3| \le \frac{1\cdot(0.1)^4}{4!} = \frac{10^{-4}}{24} \approx 0.00000417.$$

5. **(E)**. The Maclaurin series for $\arctan x$ is
$$\arctan x = \sum_{k=0}^{\infty}(-1)^k\,\frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$
The first three nonzero terms are $x - x^3/3 + x^5/5$. By the Alternating Series Estimation Theorem, the error in truncating after these three terms is bounded by the magnitude of the next term:
$$\big|\text{error}\big| \le \left|\frac{(-1)^3\,(0.5)^7}{7}\right| = \frac{(0.5)^7}{7}.$$

6. **(D)**. Remainder for $\ln(1+x)$ at $x = 0.2$, first two nonzero terms:
$$|R_2| \le \frac{|x|^3}{3} = \frac{0.008}{3} \approx 0.0027.$$

7. **(B)**. We use Taylor's Inequality for e^x about 0 with $n = 2$:
$$|R_2(x)| \le \frac{e^{|x|}\,|x|^3}{6}.$$
We test each endpoint:
$$|x| = 0.05 : \quad |R_2| \le \frac{e^{0.05}\,(0.05)^3}{6} \approx \frac{1.0513\times 0.000125}{6} \approx 2.19\times10^{-5} < 0.001,$$
$$|x| = 0.1 : \quad |R_2| \le \frac{e^{0.1}\,(0.1)^3}{6} \approx \frac{1.1052\times 0.001}{6} \approx 1.84\times10^{-4} < 0.001,$$
$$|x| = 0.5 : \quad |R_2| \le \frac{e^{0.5}\,(0.5)^3}{6} \approx \frac{1.6487\times 0.125}{6} \approx 0.0343 > 0.001.$$

8. **(A)**. For $\sin x$ centered at 0 with degree 3, Taylor's Inequality gives
$$|R_3(x)| \le \frac{|x|^4}{24}.$$
Test the candidate intervals:
$$|x| = 0.1 : \quad |R_3| \le \frac{0.1^4}{24} = \frac{10^{-4}}{24} \approx 4.17\times10^{-6} < 0.0001,$$
$$|x| = 0.5 : \quad |R_3| \le \frac{0.5^4}{24} = \frac{0.0625}{24} \approx 0.0026 > 0.0001.$$
Therefore the largest listed interval is $\boxed{|x| < 0.1}$.

9.1 Differential Equations (Solutions)

Free Response Practice

1. Verify that $y = e^{3x}$ is a solution to $y' = 3y$.

$$\begin{aligned} y &= e^{3x} \\ y' &= 3e^{3x} = 3y \end{aligned}$$

2. Verify that $y = \sin(2x)$ is a solution to $y'' + 4y = 0$.

$$\begin{aligned} y &= \sin(2x) \\ y' &= 2\cos(2x) \\ y'' &= -4\sin(2x) \\ y'' + 4y &= -4\sin(2x) + 4\sin(2x) = 0 \end{aligned}$$

3. Verify that $y = x^2 + 1$ satisfies $y' = 2x$.

$$\begin{aligned} y &= x^2 + 1 \\ y' &= 2x \end{aligned}$$

4. Solve $y' = 2x$ given $y(1) = 5$.

$$\begin{aligned} \frac{dy}{dx} &= 2x \\ \int dy &= \int 2x\,dx \\ y &= x^2 + C \\ \text{Use } y(1) = 5: \quad 5 &= 1^2 + C \Rightarrow C = 4 \\ &\boxed{y = x^2 + 4} \end{aligned}$$

5. Find the particular solution to $\frac{dy}{dx} = 3e^x$ with $y(0) = 2$.

$$\begin{aligned} \frac{dy}{dx} &= 3e^x \\ \int dy &= \int 3e^x\,dx \\ y &= 3e^x + C \\ \text{Use } y(0) = 2: \quad 2 = 3e^0 + C &= 3 + C \Rightarrow C = -1 \\ &\boxed{y = 3e^x - 1} \end{aligned}$$

6. Solve $\frac{dy}{dx} = \sin(x)$, with $y\left(\frac{\pi}{2}\right) = 0$.

$$\begin{aligned} \frac{dy}{dx} &= \sin(x) \\ \int dy &= \int \sin(x)\,dx \\ y &= -\cos(x) + C \\ \text{Use } y\left(\frac{\pi}{2}\right) = 0: \quad 0 = -\cos\left(\frac{\pi}{2}\right) &+ C \Rightarrow C = 0 \\ &\boxed{y = -\cos(x)} \end{aligned}$$

Multiple Choice Practice

1. Verify whether $y = e^{2x}$ is a solution to $y' = 2y$.

$$y = e^{2x}, \quad y' = 2e^{2x}$$
$$2y = 2e^{2x} = y'$$

(A) Yes

2. Is $y = x^2 + 1$ a solution to $y' = 2x$?

$$y = x^2 + 1, \quad y' = 2x$$

(A) Yes

3. Which of the following functions satisfy $y' = 3y$? (Select all that apply). **(A) and (B)**

4. Which of the following satisfy $y'' + y = 0$? (Select all that apply). **(A), (B), and (D)**

5. Solve $y' = 6x$.

$$\int dy = \int 6x\,dx$$
$$y = 3x^2 + C$$

(D) $y = 3x^2 + C$

6. Solve $\dfrac{dy}{dx} = \cos(x)$.

$$y = \int \cos(x)\,dx = \sin(x) + C$$

(B) $y = \sin(x) + C$

7. Solve $y' = 2x$, with $y(1) = 6$.

$$\int dy = \int 2x\,dx$$
$$y = x^2 + C$$
$$y(1) = 6 \Rightarrow 1 + C = 6 \Rightarrow C = 5$$
$$\boxed{y = x^2 + 4}$$

(D)

8. Find the particular solution to $\dfrac{dy}{dx} = e^x$, $y(0) = 3$.

$$y = \int e^x\,dx = e^x + C$$
$$y(0) = 3 \Rightarrow 1 + C = 3 \Rightarrow C = 2$$
$$\boxed{y = e^x + 2}$$

(B)

9. True or False: If $y' = 2y$, then any solution graph must be increasing wherever $y > 0$.
True: If $y > 0$, then $y' = 2y > 0$, so the graph increases.

10. True or False: If $y' = -y$, then solution curves are always decreasing when $y > 0$.
True: If $y > 0$, then $y' = -y < 0$, so the graph decreases.

9.3 Separable Differential Equations (Solutions)

Free Response Practice

1. Solve $y' = xy$

Note: $y = 0$ is a contant solution. If $y \neq 0$, then

$$\begin{aligned}\frac{dy}{dx} &= xy\\ \frac{1}{y}\,dy &= x\,dx\\ \int \frac{1}{y}\,dy &= \int x\,dx\\ \ln|y| &= \frac{1}{2}x^2 + C\\ |y| &= e^{\frac{1}{2}x^2+C} = e^C \cdot e^{x^2/2}\\ y &= \pm e^C \cdot e^{x^2/2}\\ y &= Ae^{x^2/2}\end{aligned}$$

where A is any constant, including 0.

2. Solve $y' = \frac{2y}{x}$

Note: $y = 0$ is a contant solution. If $y \neq 0$, then

$$\begin{aligned}\frac{dy}{dx} &= \frac{2y}{x}\\ \frac{1}{y}\,dy &= \frac{2}{x}\,dx\\ \int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx\\ \ln|y| &= 2\ln|x| + C\\ |y| &= e^C \cdot |x|^2\\ y &= \pm e^C \cdot x^2\\ y &= Ax^2\end{aligned}$$

where A is any constant, including 0.

3. Solve $y' = x(1+y^2)$

$$\begin{aligned}\frac{dy}{dx} &= x(1+y^2)\\ \frac{1}{1+y^2}\,dy &= x\,dx\\ \int \frac{1}{1+y^2}\,dy &= \int x\,dx\\ \arctan y &= \frac{1}{2}x^2 + C\\ y &= \tan\left(\frac{1}{2}x^2 + C\right)\end{aligned}$$

4. Solve $y' = y^2$, with $y(0) = 1$

$$\begin{aligned}\frac{dy}{dx} &= y^2\\ \frac{1}{y^2}\,dy &= dx\\ \int y^{-2}\,dy &= \int dx\\ -\frac{1}{y} &= x + C \Rightarrow y = \frac{-1}{x+C}\end{aligned}$$

$$y(0) = 1 \Rightarrow \frac{-1}{C} = 1 \Rightarrow C = -1$$

$$\boxed{y = \frac{-1}{x-1}}$$

5. Solve $y' = (1-x^2)y$, with $y(0) = 2$

$$\begin{aligned}\frac{dy}{dx} &= (1-x^2)y\\ \frac{1}{y}\,dy &= (1-x^2)\,dx\\ \int \frac{1}{y}\,dy &= \int (1-x^2)\,dx\\ \ln|y| &= x - \frac{1}{3}x^3 + C\\ |y| &= e^C \cdot e^{x-x^3/3}\\ y &= \pm e^C \cdot e^{x-x^3/3} = Ae^{x-x^3/3}\end{aligned}$$

$$y(0) = 2 \Rightarrow A = 2$$

$$\boxed{y = 2e^{x-x^3/3}}$$

6. Solve $y' = x^2(1+y)$, with $y(1) = 0$

$$\begin{aligned}\frac{dy}{dx} &= x^2(1+y)\\ \frac{1}{1+y}\,dy &= x^2\,dx\\ \int \frac{1}{1+y}\,dy &= \int x^2\,dx\\ \ln|1+y| &= \frac{1}{3}x^3 + C\\ |1+y| &= e^C \cdot e^{x^3/3}\\ 1+y &= \pm e^C \cdot e^{x^3/3} = Ae^{x^3/3}\\ y &= Ae^{x^3/3} - 1\end{aligned}$$

$$y(1) = 0 \Rightarrow Ae^{1/3} = 1 \Rightarrow A = e^{-1/3}$$

$$\boxed{y = e^{(x^3-1)/3} - 1}$$

7. Solve $y' = \frac{y}{x}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} \\
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln|y| &= \ln|x| + C \\
|y| &= e^C|x| \\
y &= \pm e^C x = Ax
\end{aligned}$$

$$\boxed{y = Ax}$$

8. Solve $y' = \frac{x}{y}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x}{y} \\
y\,dy &= x\,dx \\
\int y\,dy &= \int x\,dx \\
\frac{1}{2}y^2 &= \frac{1}{2}x^2 + C
\end{aligned}$$

$$\boxed{y^2 = x^2 + C}$$

9. Solve $y' = \frac{2y}{x+1}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2y}{x+1} \\
\frac{1}{y}\,dy &= \frac{2}{x+1}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x+1}\,dx \\
\ln|y| &= 2\ln|x+1| + C \\
|y| &= e^C|x+1|^2 \\
y &= \pm e^C(x+1)^2 = A(x+1)^2
\end{aligned}$$

$$\boxed{y = A(x+1)^2}$$

Multiple Choice Practice

1. **(C)** $\dfrac{dy}{dx} = xy$
   Separable: $\frac{1}{y}dy = x\,dx$

2. **(C)** $\dfrac{dy}{dx} = x + y$
   Not separable — cannot write as a product $g(x)h(y)$

3. **(C)** $y = Ce^{x^3/3}$
   From $\frac{dy}{dx} = x^2y \Rightarrow \frac{1}{y}dy = x^2dx \Rightarrow \ln|y| = \frac{1}{3}x^3 + C$

4. **(A)** $y = Cx^3$
   From $\frac{dy}{dx} = \frac{3y}{x} \Rightarrow \frac{1}{y}dy = \frac{3}{x}dx \Rightarrow \ln|y| = 3\ln|x| + C$

5. **(A)** $y = \dfrac{1}{2-x}$
   From $\frac{dy}{dx} = y^2 \Rightarrow -\frac{1}{y} = x + C$, use $y(0) = 2 \Rightarrow C = -\frac{1}{2}$

6. **(B)** $y = 3e^{x-x^3/3}$
   Separable: $\frac{1}{y}dy = (1-x^2)dx \Rightarrow \ln|y| = x - \frac{1}{3}x^3 + C$, then apply initial condition

7. **(B)** $y = Cx^2$
   From $\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{1}{y}dy = \frac{2}{x}dx \Rightarrow \ln|y| = 2\ln|x| + C$

8. **(A)** $y^2 = \dfrac{2x^3}{3} + C$
   From $\frac{dy}{dx} = \frac{x^2}{y} \Rightarrow y\,dy = x^2dx \Rightarrow \frac{1}{2}y^2 = \frac{2}{3}x^3 + C$