

Mixed Practice Solutions

1. Evaluate the integral: $\int \frac{2x}{x^2 + 5} dx$

Solution:

Use u -substitution:

- Let $u = x^2 + 5$, so $du = 2x dx$

$$\begin{aligned}\int \frac{2x}{x^2 + 5} dx &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |x^2 + 5| + C\end{aligned}$$

$$\int \frac{2x}{x^2 + 5} dx = \ln(x^2 + 5) + C$$

2. Evaluate the integral: $\int \ln(x) dx$

Solution:

Use integration by parts:

- Let $u = \ln(x)$, so $du = \frac{1}{x} dx$
- Let $dv = dx$, so $v = x$

Apply the formula $\int u dv = uv - \int v du$:

$$\begin{aligned}\int \ln(x) dx &= x \ln(x) - \int x \cdot \frac{1}{x} dx \\ &= x \ln(x) - \int 1 dx \\ &= x \ln(x) - x + C\end{aligned}$$

$$\int \ln(x) dx = x \ln(x) - x + C$$

3. Evaluate the integral: $\int \frac{x+1}{(x-1)(x+2)} dx$ using partial fraction decomposition.

Solution:

Step 1: Decompose the rational function.

We write:

$$\frac{x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Multiply both sides by $(x-1)(x+2)$:

$$\begin{aligned}x+1 &= A(x+2) + B(x-1) \\ &= Ax + 2A + Bx - B \\ &= (A+B)x + (2A-B)\end{aligned}$$

Match coefficients:

$$A + B = 1$$

$$2A - B = 1$$

Solving:

- Add equations: $(A+B) + (2A-B) = 1+1 \Rightarrow 3A = 2 \Rightarrow A = \frac{2}{3}$
- Then $B = 1 - A = 1 - \frac{2}{3} = \frac{1}{3}$

Step 2: Rewrite and integrate.

$$\begin{aligned}\int \frac{x+1}{(x-1)(x+2)} dx &= \int \left(\frac{2}{3(x-1)} + \frac{1}{3(x+2)} \right) dx \\ &= \frac{2}{3} \int \frac{1}{x-1} dx + \frac{1}{3} \int \frac{1}{x+2} dx \\ &= \frac{2}{3} \ln |x-1| + \frac{1}{3} \ln |x+2| + C\end{aligned}$$

$$\int \frac{x+1}{(x-1)(x+2)} dx = \frac{2}{3} \ln |x-1| + \frac{1}{3} \ln |x+2| + C$$

4. Evaluate the integral: $\int \frac{1}{x^2\sqrt{x^2-9}} dx$

Solution:

- Recognize the form $\sqrt{x^2 - a^2}$, so let:

$$x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta d\theta$$

- Substitute into the integral:

$$\begin{aligned} \int \frac{1}{x^2\sqrt{x^2-9}} dx &= \int \frac{1}{9 \sec^2 \theta \cdot 3 \tan \theta} \cdot 3 \sec \theta \tan \theta d\theta \\ &= \int \frac{3 \sec \theta \tan \theta}{27 \sec^2 \theta \tan \theta} d\theta \\ &= \int \frac{3}{27 \sec \theta} d\theta \\ &= \int \frac{1}{9 \sec \theta} d\theta \\ &= \int \frac{\cos \theta}{9} d\theta \\ &= \frac{1}{9} \int \cos \theta d\theta \\ &= \frac{1}{9} \sin \theta + C \end{aligned}$$

- Convert back to x :

$$\sec \theta = \frac{x}{3} \Rightarrow \cos \theta = \frac{3}{x}, \quad \sin \theta = \frac{\sqrt{x^2-9}}{x}$$

- Final answer:

$$\boxed{\int \frac{1}{x^2\sqrt{x^2-9}} dx = \frac{\sqrt{x^2-9}}{9x} + C}$$

5. Evaluate the integral: $\int x e^{x^2} dx$ using an appropriate method.

Solution:

Use u -substitution:

- Let $u = x^2$, so $du = 2x dx \Rightarrow \frac{1}{2} du = x dx$

$$\begin{aligned} \int x e^{x^2} dx &= \int e^u \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{x^2} + C \end{aligned}$$

$$\boxed{\int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C}$$

6. Find the slope $\frac{dy}{dx}$ of the parametric curve at $t = \pi/6$, where $x(t) = \sin(2t)$ and $y(t) = \cos(t)$

Solution:

We use the chain rule for parametric equations:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- Compute derivatives:

$$\frac{dx}{dt} = \frac{d}{dt}[\sin(2t)] = 2 \cos(2t), \quad \frac{dy}{dt} = \frac{d}{dt}[\cos(t)] = -\sin(t)$$

- Plug in $t = \frac{\pi}{6}$:

$$\cos(2t) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin(t) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

- Compute:

$$\frac{dy}{dx} = \frac{-\sin(t)}{2 \cos(2t)} = \frac{-\frac{1}{2}}{2 \cdot \frac{1}{2}} = \frac{-1/2}{1} = -\frac{1}{2}$$

$$\boxed{\frac{dy}{dx} = -\frac{1}{2}}$$

7. Find the average value of the function $f(x) = \sqrt{x}$ on the interval $[1, 4]$.

Solution:

The average value of a function on $[a, b]$ is given by:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$

- In this case, $a = 1$, $b = 4$, and $f(x) = \sqrt{x}$

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{4-1} \int_1^4 \sqrt{x} dx \\ &= \frac{1}{3} \int_1^4 x^{1/2} dx \\ &= \frac{1}{3} \left[\frac{2}{3} x^{3/2} \right]_1^4 \\ &= \frac{1}{3} \cdot \frac{2}{3} (4^{3/2} - 1^{3/2}) \\ &= \frac{2}{9} (8 - 1) \\ &= \frac{2}{9} \cdot 7 = \frac{14}{9} \end{aligned}$$

Average value = $\frac{14}{9}$

8. Find the volume of the solid with base bounded by $y = x^2$ and $y = 4$, where cross-sections perpendicular to the x -axis are semicircles.

Solution:

- The region is bounded between $x = -2$ and $x = 2$ because $x^2 = 4 \Rightarrow x = \pm 2$.
- At each x , the vertical distance between the curves is the diameter of the semicircle:

$$\text{diameter} = 4 - x^2 \quad \Rightarrow \quad \text{radius} = \frac{4 - x^2}{2}$$

- Area of a semicircle:

$$A(x) = \frac{1}{2} \pi \left(\frac{4 - x^2}{2} \right)^2 = \frac{\pi}{8} (4 - x^2)^2$$

- Volume is given by:

$$V = \int_{-2}^2 A(x) dx = \int_{-2}^2 \frac{\pi}{8} (4 - x^2)^2 dx$$

Now expand and integrate:

$$\begin{aligned} V &= \frac{\pi}{8} \int_{-2}^2 (16 - 8x^2 + x^4) dx \\ &= \frac{\pi}{8} \left[\int_{-2}^2 16 dx - \int_{-2}^2 8x^2 dx + \int_{-2}^2 x^4 dx \right] \\ &= \frac{\pi}{8} \left[16x \Big|_{-2}^2 - 8 \cdot \frac{x^3}{3} \Big|_{-2}^2 + \frac{x^5}{5} \Big|_{-2}^2 \right] \\ &= \frac{\pi}{8} \left(64 - \frac{128}{3} + \frac{64}{5} \right) \\ &= \frac{64\pi}{15} \end{aligned}$$

9. Set up an integral representing the volume of the solid formed by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the x -axis.

Solution:

- This is a rotation about the x -axis, so we use the **disk method**.
- The radius of a representative disk is $y = \sqrt{x}$, so the area of a cross-sectional disk is:

$$A(x) = \pi(\sqrt{x})^2 = \pi x$$

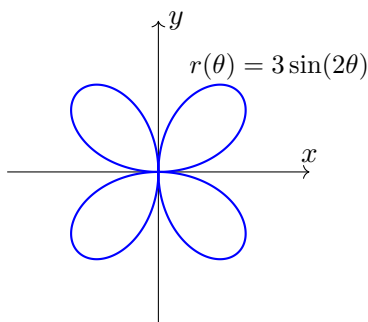
- The volume is:

$$V = \int_0^4 \pi x \, dx$$

10. Set up, but do not evaluate, an integral representing the area enclosed by one loop of the curve:

$$r(\theta) = 3 \sin(2\theta)$$

Solution:



- The general formula for area in polar coordinates is:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [r(\theta)]^2 \, d\theta$$

- Since $r(\theta) = 3 \sin(2\theta)$, one loop occurs as θ goes from 0 to $\frac{\pi}{2}$.
- Thus, the area of one loop is:

$$A = \frac{1}{2} \int_0^{\frac{\pi}{2}} [3 \sin(2\theta)]^2 \, d\theta$$

11. Find the exact sum of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

by recognizing it as a telescoping series.

Solution:

Step 1: Use partial fraction decomposition.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Step 2: Write the partial sum S_n :

$$\begin{aligned} S_n &= \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \end{aligned}$$

Step 3: Simplify the telescoping sum.

All interior terms cancel:

$$S_n = 1 - \frac{1}{n+1}$$

Step 4: Take the limit as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

12. Determine whether the following series converges or diverges: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+2}$

Solution:

This is an alternating series of the form:

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad \text{where } b_n = \frac{1}{n+2}$$

To apply the Alternating Series Test, check:

- $b_n = \frac{1}{n+2} > 0$
- b_n is decreasing: $\frac{1}{n+2} > \frac{1}{n+3}$ for all $n \geq 1$
- $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{n+2} = 0$

Conclusion: All conditions of the Alternating Series Test are satisfied, so the series converges by the Alternating Series Test.

13. Find the radius of convergence and interval of convergence for the series:

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n2^n}$$

Solution:

We apply the Ratio Test:

$$\begin{aligned} a_n &= \frac{(x+2)^n}{n \cdot 2^n} \\ a_{n+1} &= \frac{(x+2)^{n+1}}{(n+1) \cdot 2^{n+1}} \end{aligned}$$

Step 1: Compute the limit:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{(n+1) \cdot 2^{n+1}} \cdot \frac{n \cdot 2^n}{(x+2)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(x+2) \cdot n}{2(n+1)} \right| \\ &= \left| \frac{x+2}{2} \right| \cdot \lim_{n \rightarrow \infty} \frac{n}{n+1} = \left| \frac{x+2}{2} \right| \end{aligned}$$

The series converges when this limit is less than 1:

$$\left| \frac{x+2}{2} \right| < 1 \quad \Rightarrow \quad |x+2| < 2$$

Radius of convergence:

$$\boxed{R = 2}$$

Step 2: Find the interval of convergence. We test the endpoints $x = -4$ and $x = 0$.

At $x = -4$:

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Rightarrow \text{converges}$$

At $x = 0$:

$$\sum_{n=1}^{\infty} \frac{2^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \text{diverges}$$

Final Answer:

$$\boxed{\text{Radius of convergence: } R = 2}$$

$$\boxed{\text{Interval of convergence: } [-4, 0)}$$

14. Find the first four nonzero terms of the Taylor series for $f(x) = e^{-x^2}$ centered at $a = 0$.

Solution:

Recall the Taylor series for e^x centered at 0:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To find the Taylor series for e^{-x^2} , substitute $-x^2$ in place of x :

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Now write out the first four nonzero terms:

$$\begin{aligned} e^{-x^2} &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots \end{aligned}$$

$$\boxed{e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots}$$

15. Find the third-degree Taylor polynomial centered at $a = 0$ for $f(x) = \sin(2x)$.

Solution:

We need the terms up to degree 3 in the Taylor series for $\sin(2x)$ about $x = 0$.

- Recall the Taylor series for $\sin(x)$ centered at 0:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

- Substitute $2x$ in place of x :

$$\begin{aligned} \sin(2x) &= 2x - \frac{(2x)^3}{3!} + \cdots \\ &= 2x - \frac{8x^3}{6} + \cdots \\ &= 2x - \frac{4x^3}{3} + \cdots \end{aligned}$$

$$\boxed{P_3(x) = 2x - \frac{4x^3}{3}}$$

16. Find the third degree Taylor polynomial for $f(x) = \cos x$ centered at $a = \pi/6$. Use Taylor's Inequality to give an upper bound on the error if this approximation is used on the interval $0 < x < \pi/3$.

Solution:

Step 1: Compute derivatives of $f(x) = \cos x$:

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \quad (\text{and so on}) \end{aligned}$$

Step 2: Evaluate derivatives at $a = \pi/6$:

$$\begin{aligned} f\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \\ f'\left(\frac{\pi}{6}\right) &= -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2} \\ f''\left(\frac{\pi}{6}\right) &= -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} \\ f^{(3)}\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \end{aligned}$$

Step 3: Write the 3rd-degree Taylor polynomial centered at $a = \pi/6$:

$$\begin{aligned} P_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3 \end{aligned}$$

$$P_3(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3$$

Step 4: Use Taylor's Inequality to bound the error on $0 < x < \pi/3$.

We need an upper bound for:

$$|R_3(x)| \leq \frac{M}{4!}|x-a|^4$$

where M is a bound on $|f^{(4)}(x)| = |\cos x|$ on the interval $0 < x < \pi/3$. Since $|\cos x| \leq 1$, we take $M = 1$.

Also, the maximum value of $|x - \pi/6|$ on the interval $(0, \pi/3)$ is:

$$\max \left| x - \frac{\pi}{6} \right| = \frac{\pi}{6}$$

$$|R_3(x)| \leq \frac{1}{4!} \left(\frac{\pi}{6} \right)^4$$

17. Solve the differential equation:

$$\frac{dy}{dx} = xy^2, \quad y(0) = 1$$

Solution:

Step 1: Separate the variables.

$$\begin{aligned} \frac{dy}{dx} &= xy^2 \\ \Rightarrow \frac{1}{y^2} dy &= x dx \end{aligned}$$

Step 2: Integrate both sides.

$$\begin{aligned} \int \frac{1}{y^2} dy &= \int x dx \\ \int y^{-2} dy &= \int x dx \\ -y^{-1} &= \frac{x^2}{2} + C \end{aligned}$$

Step 3: Solve for y .

$$\begin{aligned} -\frac{1}{y} &= \frac{x^2}{2} + C \\ \Rightarrow y &= \frac{-1}{\frac{x^2}{2} + C} \end{aligned}$$

Step 4: Apply the initial condition $y(0) = 1$.

$$y(0) = \frac{-1}{0+C} = 1 \quad \Rightarrow \quad C = -1$$

Final Answer:

$$y(x) = \frac{-1}{\frac{x^2}{2} - 1} = \frac{-2}{x^2 - 2}$$